APPLIED DIFFERENTIAL GEOMETRY

A Modern Introduction

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A Modern Introduction

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Dedicated to:

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Preface

Applied Differential Geometry: A Modern Introduction is a graduate–level monographic textbook. It is designed as a comprehensive introduction into methods and techniques of modern differential geometry with its various physical and non–physical applications. In some sense, it is a continuation of our previous book, *Natural Biodynamics* (World Scientific, 2006), which contains all the necessary background for comprehensive reading of the current book. While the previous book was focused on biodynamic applications, the core applications of the new book are in the realm of modern theoretical physics, mainly following its central line: *Einstein–Feynman–Witten.* Other applications include (among others): control theory, robotics, neurodynamics, psychodynamics and socio–economical dynamics.

The book has six chapters. Each chapter contains both 'pure mathematics' and related 'applications' labelled by the word 'APPLICATION'.

The first chapter provides a soft ('plain–English') introduction into manifolds and related geometrical structures, for all the interested readers without the necessary background. As a 'snap–shot' illustration, at the end of the first chapter, a paradigm of generic differential–geometric modelling is given, which is supposed to fit all above–mentioned applications.

The second chapter gives technical preliminaries for development of the modern applied differential geometry. These preliminaries include: (i) classical geometrical objects – *tensors*, (ii) both classical and modern physical objects – *actions*, and modern geometrical objects – *functors*.

The third chapter develops modern *manifold geometry*, together with its main physical and non–physical applications. This chapter is a neccessary background for comprehensive reading of the remaining chapters.

The fourth chapter develops modern *bundle geometry*, together with its main physical and non–physical applications.

The fifth chapter develops modern *jet bundle geometry*, together with its main applications in non–autonomous mechanics and field physics. All material in this chapter is based on the previous chapter.

The sixth chapter develops modern geometrical machinery of Feynman's *path integrals*, together with their various physical and non–physical applications. For most of this chapter, only the third chapter is a neccessary background, assuming a basic understanding of quantum mechanics (as provided in the above–mentioned World Scientific book, *Natural Biodynamics*).

The book contains both an extensive Index (which allows easy connections between related topics) and a number of cited references related to modern applied differential geometry.

Our approach to dynamics of complex systems is somewhat similar to the approach to mathematical physics used at the beginning of the 20th Century by the two leading mathematicians: David Hilbert and John von Neumann – the approach of combining mathematical rigor with conceptual clarity, or *geometrical intuition* that underpins the rigor.

The intended audience includes (but is not restricted to) theoretical and mathematical physicists; applied and pure mathematicians; control, robotics and mechatronics engineers; computer and neural scientists; mathematically strong chemists, biologists, psychologists, sociologists and economists – both in academia and industry.

Compared to all differential–geometric books published so far, *Applied Differential Geometry: A Modern Introduction* has much wider variety of both physical and non–physical applications. After comprehensive reading of this book, a reader should be able to both *read* and *write* journal papers in such diverse fields as superstring & topological quantum field theory, nonlinear dynamics & control, robotics, biomechanics, neurodynamics, psychodynamics and socio–economical dynamics.

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Glossary of Frequently Used Symbols

General

– ‘iff’ means ‘if and only if’;
– ‘r.h.s’ means ‘right hand side’; ‘l.h.s’ means ‘left hand side’;
– ODE means ordinary differential equation, while PDE means partial differential equation;
– *Einstein’s summation convention over repeated indices* (not necessarily one up and one down) *is assumed in the whole text,* unless explicitly stated otherwise.

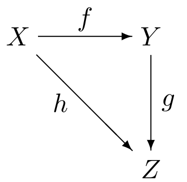
Sets

$\mathbb{N}$ – natural numbers;
$\mathbb{Z}$ – integers;
$\mathbb{R}$ – real numbers;
$\mathbb{C}$ – complex numbers;
$\mathbb{H}$ – quaternions;
$\mathbb{K}$ – number field of real numbers, complex numbers, or quaternions.

Maps

$f : A \to B$ – a function, (or map) between sets $A \equiv \operatorname{Dom} f$ and $B \equiv \operatorname{Cod} f$;

$$\operatorname{Ker} f = f^{-1}(e_B) - \text{a kernel of } f;$$
$$\operatorname{Im} f = f(A) - \text{an image of } f;$$
$$\operatorname{Coker} f = \operatorname{Cod} f / \operatorname{Im} f - \text{a cokernel of } f;$$
$$\operatorname{Coim} f = \operatorname{Dom} f / \operatorname{Ker} f - \text{a coimage of } f;$$



– a commutative diagram, requiring $h = g \circ f$.

Derivatives

$C^k(A, B)$ – set of $k-$times differentiable functions between sets A to B;
$C^\infty(A, B)$ – set of *smooth* functions between sets A to B;
$C^0(A, B)$ – set of *continuous* functions between sets A to B;
$f'(x) = \frac{df(x)}{dx}$ – derivative of f with respect to x;
$\dot{x}$ – total time derivative of x;
$\partial_t \equiv \frac{\partial}{\partial t}$ – partial time derivative;
$\partial_{x^i} \equiv \partial_i \equiv \frac{\partial}{\partial x^i}$ – partial coordinate derivative;
$\dot{f} = \partial_t f + \partial_{x^i} f \, \dot{x}^i$ – total time derivative of the scalar field $f = f(t, x^i)$;
$u_t \equiv \partial_t u, \; u_x \equiv \partial_x u, \; u_{xx} \equiv \partial_{x^2} u$ – only in partial differential equations;
$L_{x^i} \equiv \partial_{x^i} L, \; L_{\dot{x}^i} \equiv \partial_{\dot{x}^i} L$ – coordinate and velocity partial derivatives of the Lagrangian function;
d – exterior derivative;
d^n – coboundary operator;
∂_n – boundary operator;
$\nabla = \nabla(g)$ – affine Levi–Civita connection on a smooth manifold M with Riemannian metric tensor $g = g_{ij}$;
Γ^i_{jk} – Christoffel symbols of the affine connection ∇;
$\nabla_X T$ – covariant derivative of the tensor–field T with respect to the vector–field X, defined by means of Γ^i_{jk};
$T_{;x^i} \equiv T_{|x^i}$ – covariant derivative of the tensor–field T with respect to the coordinate basis $\{x^i\}$;
$\dot{\bar{T}} \equiv \frac{DT}{dt} \equiv \frac{\nabla T}{dt}$ – absolute (intrinsic, or Bianchi) derivative of the tensor–field T upon the parameter t; e.g., acceleration vector is the *absolute time derivative* of the velocity vector, $a^i = \dot{\bar{v}}^i \equiv \frac{Dv^i}{dt}$; note that in general, $a^i \neq \dot{v}^i$ – this is crucial for *proper definition of Newtonian force*;
$\mathcal{L}_X T$ – Lie derivative of the tensor–field T in direction of the vector–field X;
$[X, Y]$ – Lie bracket (commutator) of two vector–fields X and Y;
$[F, G]$, or $\{F, G\}$ – Poisson bracket, or Lie–Poisson bracket, of two functions F and G.

Smooth Manifolds, Fibre Bundles and Jet Spaces

Unless otherwise specified, all *manifolds* $M, N, ...$ are assumed C^k−smooth, real, finite–dimensional, Hausdorff, paracompact, connected and without boundary,[1] while all maps are assumed C^k−smooth. We use the symbols $\otimes$, $\vee$, $\wedge$ and $\oplus$ for the tensor, symmetrized and exterior products, as well as the Whitney sum[2], respectively, while $\rfloor$ denotes the interior product (contraction) of (multi)vectors and p−forms, and $\hookrightarrow$ denotes a manifold imbedding (i.e., both a submanifold and a topological subspace of the codomain manifold). The symbols ∂^A_B denote partial derivatives with respect to coordinates possessing multi–indices ^B_A (e.g., $\partial_\alpha = \partial/\partial x^\alpha$);
TM – tangent bundle of the manifold M;
$\pi_M : TM \to M$ – natural projection;
T^*M – cotangent bundle of the manifold M;
$\pi : Y \to X$ – fibre bundle;
(E, π, M) – vector bundle with total space E, base M and projection π;
(Y, π, X, V) – fibre bundle with total space Y, base X, projection π and standard fibre V;
$J^k(M, N)$ – space of k−jets of smooth functions between manifolds M and N;
$J^k(X, Y)$ – k−-jet space of a fibre bundle $Y \to X$; in particular, in mechanics we have a 1–jet space $J^1(\mathbb{R}, Q)$, with 1–jet coordinate maps $j^1_t s : t \mapsto (t, x^i, \dot{x}^i)$, as well as a 2–jet space $J^2(\mathbb{R}, Q)$, with 2–jet coordinate maps $j^2_t s : t \mapsto (t, x^i, \dot{x}^i, \ddot{x}^i)$;
$j^k_x s$ – k−jets of sections $s^i : X \to Y$ of a fibre bundle $Y \to X$;
We use the following kinds of *manifold maps*: immersion, imbedding, submersion, and projection. A map $f : M \to M'$ is called the *immersion* if the tangent map Tf at every point $x \in M$ is an injection (i.e., '1–1' map). When f is both an immersion and an injection, its image is said to be a submanifold of M'. A submanifold which also is a topological subspace is called imbedded submanifold. A map $f : M \to M'$ is called *submersion* if the tangent map Tf at every point $x \in M$ is a surjection (i.e., 'onto' map). If f is both a submersion and a surjection, it is called *projection* or *fibre bundle.*

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[1] The only 1D manifolds obeying these conditions are the real line $\mathbb{R}$ and the circle S^1.

[2] *Whitney sum* $\oplus$ is an analog of the direct (Cartesian) product for vector bundles. Given two vector bundles Y and Y' over the same base X, their Cartesian product is a vector bundle over $X \times X$. The *diagonal map* induces a vector bundle over X called the Whitney sum of these vector bundles and denoted by $Y \oplus Y'$.

Lie and (Co)Homology Groups

G – usually a general Lie group;
$GL(n)$ – general linear group with real coefficients in dimension n;
$SO(n)$ – group of rotations in dimension n;
T^n – toral (Abelian) group in dimension n;
$Sp(n)$ – symplectic group in dimension n;
$T(n)$ – group of translations in dimension n;
$SE(n)$ – Euclidean group in dimension n;
$H_n(M) = \mathrm{Ker}\,\partial_n / \mathrm{Im}\,\partial_{n-1}$ – nth homology group of the manifold M;
$H^n(M) = \mathrm{Ker}\,d^n / \mathrm{Im}\,d^{n+1}$ – nth cohomology group of the manifold M.

Other Spaces and Operators

$\mathrm{i} \equiv \sqrt{-1}$ – imaginary unit;
$C^k(M)$ – space of $k-$differentiable functions on the manifold M;
$\Omega^k(M)$ – space of $k-$forms on the manifold M;
$\mathfrak{g}$ – Lie algebra of a Lie group G, i.e., the tangent space of G at its identity element;
$Ad(g)$ – adjoint endomorphism; recall that *adjoint representation* of a Lie group G is the linearized version of the action of G on itself by conjugation, i.e., for each $g \in G$, the inner automorphism $x \mapsto gxg^{-1}$ gives a linear transformation $Ad(g) : \mathfrak{g} \to \mathfrak{g}$, from the Lie algebra $\mathfrak{g}$ of G to itself;
nD space (group, system) means $n-$dimensional space (group, system), for $n \in \mathbb{N}$;
$\rhd$ – semidirect (noncommutative) product; e.g., $SE(3) = SO(3) \rhd \mathbb{R}^3$;
$\lrcorner$ – interior product, or contraction, of a vector–field and a one–form;
$\sum\!\!\!\!\!\!\int$ – Feynman path integral symbol, denoting integration over continuous spectrum of smooth paths and summation over discrete spectrum of Markov chains; e.g., $\sum\!\!\!\!\!\!\int \mathcal{D}[x]\, \mathrm{e}^{\mathrm{i}S[x]}$ denotes the path integral (i.e., sum–over–histories) over all possible paths $x^i = x^i(t)$ defined by the Hamilton action, $S[x] = \frac{1}{2}\int_{t_0}^{t_1} g_{ij}\, \dot{x}^i \dot{x}^j\, dt$, while $\sum\!\!\!\!\!\!\int \mathcal{D}[\Phi]\, \mathrm{e}^{\mathrm{i}S[\Phi]}$ denotes the path integral over all possible fields $\Phi^i = \Phi^i(x)$ defined by some field action $S[\Phi]$.

Categories

$\mathcal{S}$ – all sets as objects and all functions between them as morphisms;
$\mathcal{PS}$ – all pointed sets as objects and all functions between them preserving

base point as morphisms;
$\mathcal{V}$ – all vector spaces as objects and all linear maps between them as morphisms;
$\mathcal{B}$ – Banach spaces over $\mathbb{R}$ as objects and bounded linear maps between them as morphisms;
$\mathcal{G}$ – all groups as objects, all homomorphisms between them as morphisms;
$\mathcal{A}$ – Abelian groups as objects, homomorphisms between them as morphisms;
$\mathcal{AL}$ – all algebras (over a given number field $\mathbb{K}$) as objects, all their homomorphisms between them as morphisms;
$\mathcal{T}$ – all topological spaces as objects, all continuous functions between them as morphisms;
$\mathcal{PT}$ – pointed topological spaces as objects, continuous functions between them preserving base point as morphisms;
$\mathcal{TG}$ – all topological groups as objects, their continuous homomorphisms as morphisms;
$\mathcal{M}$ – all smooth manifolds as objects, all smooth maps between them as morphisms;
$\mathcal{M}_n$ – nD manifolds as objects, their local diffeomorphisms as morphisms;
$\mathcal{LG}$ – all Lie groups as objects, all smooth homomorphisms between them as morphisms;
$\mathcal{LAL}$ – all Lie algebras (over a given field $\mathbb{K}$) as objects, all smooth homomorphisms between them as morphisms;
$\mathcal{TB}$ – all tangent bundles as objects, all smooth tangent maps between them as morphisms;
$\mathcal{T}^*\mathcal{B}$ – all cotangent bundles as objects, all smooth cotangent maps between them as morphisms;
$\mathcal{VB}$ – all smooth vector bundles as objects, all smooth homomorphisms between them as morphisms;
$\mathcal{FB}$ – all smooth fibre bundles as objects, all smooth homomorphisms between them as morphisms;
Symplec – all symplectic manifolds (i.e., physical phase–spaces), all symplectic maps (i.e., canonical transformations) between them as morphisms;
Hilbert – all Hilbert spaces and all unitary operators as morphisms.

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