Aspects of Semidefinite Programming

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Aspects of Semidefinite Programming

Interior Point Algorithms and Selected Applications

by

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Foreword

This monograph has grown from my PhD thesis *Interior point Methods for Semidefinite Programming* [39] which was published in December 1997. Since that time, Semidefinite Programming (SDP) has remained a popular research topic and the associated body of literature has grown considerably. As SDP has proved such a useful tool in many applications, like systems and control theory and combinatorial optimization, there is a growing number of people who would like to learn more about this field.

My goal with this monograph is to provide a personal view on the theory and applications of SDP in such a way that the reader will be equipped to read the relevant research literature and explore new avenues of research. Thus I treat a selected number of topics in depth, and provide references for further reading. The chapters are structured in such a way that the monograph can be used for a graduate course on SDP.

With regard to algorithms, I have focused mainly on methods involving the so-called Nesterov–Todd (NT) direction in some way. As for applications, I have selected interesting ones — mainly in combinatorial optimization — that are not extensively covered in the existing review literature.

In making these choices I hasten to acknowledge that much of the algorithmic analysis can be done in a more general setting (*i.e.*, working with self-concordant barriers, self-dual cones and Euclidean Jordan algebras). I only consider real symmetric positive semidefinite matrix variables in this book; this already allows a wealth of applications.

Required background

The reader is expected to have some background on the following topics:

- linear algebra and multivariate calculus;
- linear and non-linear programming;
- basic convex analysis;
- combinatorial optimization or complexity theory.

I have provided appendices on the necessary matrix analysis (Appendix A), matrix calculus (Appendix C) and convex analysis & optimization (Appendix B), in an attempt to make this volume self–contained to a large degree. Nevertheless, when reading this book it will be handy to have access to the following textbooks, as I refer to them frequently.

- *Convex Analysis* by Rockafellar [160];
- *Matrix Analysis* and *Topics in Matrix Analysis* by Horn and Johnson [85, 86];
- *Nonlinear Programming: Theory and Algorithms* by Bazarraa, Sherali and Shetty [16];
- *Theory and Algorithms for Linear Optimization: An interior point approach,* by Roos, Terlaky and Vial [161];
- *Randomized Algorithms* by Motwani and Raghavan [128].

In particular, the analysis of interior point algorithms presented in this monograph owes much to the analysis in the book by Roos, Terlaky and Vial [161].

Moreover, I do not give an introduction to complexity theory or to randomized algorithms; the reader is referred to the excellent text by Motwani and Raghavan [128] for such an introduction.

List of Notation

Matrix notation

A^T	:	transpose of $A \in \mathbf{R}^{m\times n}$;
A^{-T}	:	$\left(A^T\right)^{-1}$;
a_{ij}	:	ijth entry of $A \in \mathbf{R}^{m\times n}$;
$A \sim B$	:	$A = T^{-1}BT$ for some nonsingular T
	$\equiv$	the matrices A and B are similar;
$A \succeq B$ $(A \succ B)$	:	$A - B$ is symmetric positive semidefinite (positive definite);
$A \preceq B$ $(A \prec B)$	:	$A - B$ is symmetric negative semidefinite (negative definite);
$\mathcal{R}(A)$	:	range (column space) of $A \in \mathbf{R}^{n\times n}$;

Special vectors and matrices

I_r	:	$r \times r$ identity matrix;
I	:	identity matrix of size depending on the context;
$0_{m\times n}$	:	$m \times n$ zero matrix;
0_n	:	zero vector in $\mathbf{R}^n$;
0	:	zero vector/matrix of size depending on the context;
e_n	:	vector of all-ones in $\mathbf{R}^n$;
e	:	vector of all-ones of size depending on the context;

Sets of matrices and vectors

$\mathbf{R}^n$	:	n-dimensional real Euclidian vector space;
$\mathbf{R}^n_+$	:	positive orthant of $\mathbf{R}^n$;
$\mathbb{Z}^n$	:	n-tuples of integers
$\mathbb{Z}^n_+$	:	n-tuples of *nonnegative* integers
$\mathbf{R}^{n\times n}$	:	space of real $(n \times n)$ matrices;
$\mathcal{S}_n$	=	$\{X \mid X \in \mathbf{R}^{n\times n},\ X = X^T\}$;
$\mathcal{S}_n^+$	=	$\{X \mid X \in \mathcal{S}_n,\ X \succeq 0\}$;
$\mathcal{S}_n^{++}$	=	$\{X \mid X \in \mathcal{S}_n,\ X \succ 0\}$;
$\mathcal{C}_n$	=	$\left\{A \in \mathcal{S}_n \mid x^T A x \geq 0 \quad \forall x \in \mathbf{R}^n_+\right\}$ (Copositive matrices);
$\mathcal{N}_n$	=	$\{A \in \mathcal{S}_n \mid a_{ij} \geq 0 \quad \forall i,j = 1,\ldots,n\}$ (Nonnegative matrices);
$\mathbf{svec}\left(\mathcal{S}_n^+\right)$	=	$\left\{x \in \mathbf{R}^{\frac{1}{2}n(n+1)} \,\middle|\, \mathbf{smat}(x) \in \mathcal{S}_n^+\right\}$;
$\{-1,1\}^n$	=	$\{x \in \mathbf{R}^n \mid x_i \in \{-1,1\},\ i = 1,\ldots,n\}$;

Functions of matrices

$$
\begin{array}{rcl}
\lambda_i(A) & : & i\text{th largest eigenvalue of } A, \text{ if } \lambda_j(A) \in \mathbf{R} \quad \forall j; \\
\lambda_{\max}(A) & = & \max_i \lambda_i(A), \text{ if } \lambda_i(A) \in \mathbf{R} \quad \forall i; \\
\lambda_{\min}(A) & = & \min_i \lambda_i(A), \text{ if } \lambda_i(A) \in \mathbf{R} \quad \forall i; \\
\mathbf{Tr}\,(A) & = & \sum_i a_{ii} = \sum_i \lambda_i(A) \quad (\text{trace of } A \in \mathbf{R}^{n\times n}); \\
\langle A, B\rangle & = & \mathbf{Tr}\,\left(AB^T\right); \\
\det(A) & : & \text{determinant of } A \in \mathbf{R}^{n\times n} = \prod_i \lambda_i(A); \\
\|A\|^2 & = & \mathbf{Tr}\,(AA^T) = \sum_i \sum_j a_{ij}^2 \quad (\text{Frobenius norm}) \\
 & = & \sum_i \lambda_i^2(A) \text{ if } A \in \mathcal{S}_n; \\
\|A\|_2 & = & (\lambda_{\max}(A^T A))^{\frac{1}{2}} \quad (\text{spectral norm}) \\
 & = & \lambda_{\max}(A) \text{ if } A \succeq 0; \\
\rho(A) & = & \max_i |\lambda_i(A)| \quad (\text{spectral radius of } A); \\
\kappa(A) & = & \dfrac{\lambda_{\max}(A)}{\lambda_{\min}(A)} \text{ if } \lambda_i(A) > 0 \ \forall i \\
 & = & \text{condition number of } A \text{ if } A \succ 0; \\
A^{\frac{1}{2}} & : & \text{unique symmetric square root factor of } A \succeq 0; \\
\mathrm{Diag}\,(x) & : & n \times n \text{ diagonal matrix with components of } x \in \mathbf{R}^n \text{ on diagonal}; \\
\mathrm{diag}\,(X) & : & n\text{-vector obtained by extracting diagonal of } X \in \mathbf{R}^{n\times n}; \\
\mathbf{vec}(A) & = & [a_{11},\ a_{21}, \ldots a_{n1},\ a_{12},\ a_{22}, \ldots, a_{nn}]^T \text{ for } A \in \mathbf{R}^{n\times n}; \\
\mathbf{svec}(A) & = & [a_{11},\ \sqrt{2}a_{12},\ \ldots\ \sqrt{2}a_{1n},\ a_{22},\ \sqrt{2}a_{23},\ \ldots,\ a_{nn}]^T \text{ for } A \in \mathcal{S}_n; \\
\mathbf{smat} & : & \text{inverse operator of } \mathbf{svec};
\end{array}
$$

Set notation

$$
\begin{array}{rcl}
\mathrm{ri}\,(\mathcal{C}) & : & \text{relative interior of a convex set } C; \\
\dim(\mathcal{L}) & : & \text{dimension of a subspace } \mathcal{L}; \\
\mathcal{C}^* & : & \text{Dual cone of a cone } \mathcal{C} \subset \mathbf{R}^n \\
 & = & \left\{x \in \mathbf{R}^n \mid x^T y \geq 0 \ \forall y \in \mathcal{C}\right\};
\end{array}
$$

SDP problems in standard form

(P) : primal problem in standard from;
(D) : Lagrangian dual problem of (P);
$\mathcal{P}$: feasible set of problem (P);
$\mathcal{D}$: feasible set of problem (D);
$\mathcal{P}^*$: optimal set of problem (P);
$\mathcal{D}^*$: optimal set of problem (D);
(P_{gf}) : ELSD dual of (D);
(D_{cor}) : Lagrangian dual of (P_{gf});
$\mathcal{L} = \text{span}\{A_1, \ldots, A_m\}$;

Interior point analysis

$\log(t)$: natural logarithm of $t > 0$;

$$\psi(t) = t - \log(1+t) \quad t > -1;$$

$$f_p^\mu(X) = \frac{1}{\mu}\mathbf{Tr}\,(CX) - \log\det X;$$

$$f_d^\mu(S, y) = \frac{1}{\mu}b^T y + \log\det(S);$$

$$f_{pd}^\mu(X, S) = \mathbf{Tr}\left(\frac{XS}{\mu}\right) - \log\det\left(\frac{XS}{\mu}\right) - n;$$

$$D = \left[X^{\frac{1}{2}}\left(X^{\frac{1}{2}}SX^{\frac{1}{2}}\right)^{-\frac{1}{2}}X^{\frac{1}{2}}\right]^{\frac{1}{2}} \quad \text{where } (X, S) \in \text{ri}\,(\mathcal{P} \times \mathcal{D})$$

$\equiv$ Nesterov–Todd scaling matrix;

$$V = D^{-\frac{1}{2}}XD^{-\frac{1}{2}} = D^{\frac{1}{2}}SD^{\frac{1}{2}};$$

$$\delta(X, S, \mu) = \frac{1}{2}\left\|\sqrt{\mu}V^{-1} - \frac{1}{\sqrt{\mu}}V\right\|;$$

$$D_X = D^{-\frac{1}{2}}\Delta X D^{-\frac{1}{2}} \quad (\Delta X \in \mathcal{L}^\perp);$$

$$D_S = D^{\frac{1}{2}}\Delta S D^{\frac{1}{2}} \quad (\Delta S \in \mathcal{L});$$

$$\Psi(X, S) = -\log\det(XS) + n\log\mathbf{Tr}\,(XS) - n\log n \text{ (Centrality function)};$$

$$\Phi(X, S) = (n + \nu\sqrt{n})\log\mathbf{Tr}\,(XS) - \log\det(XS) - n\log n$$

$\equiv$ Tanabe–Todd–Ye potential function;

Graph theory

$G = (V, E)$: simple undirected graph with vertex set V and edge set E;
$\alpha(G)$: stability number of G;

$\chi(G)$	:	chromatic number of G;
$\omega(G)$	:	clique number of G;
$\bar{G}$	:	complementary graph (complement) of G;
$\vartheta(G)$	:	Lovász ϑ-function of G;
$\Theta(G)$	:	Shannon capacity of G;

Notation for asymptotic analysis Let $f(n), g(n) : \mathbf{R} \mapsto \mathbf{R}_+$. We say that

$f(n) = O(g(n))$	:	$f(n)/g(n)$ is bounded from above;
$f(n) = \tilde{O}(g(n))$	:	$f(n) = O\left([\log(n)]^p\, g(n)\right)$ for some $p > 0$ (independent of n);
$f(n) = \Omega(g(n))$	:	$g(n)/f(n)$ is bounded from above;
$f(n) \sim g(n)$		$\lim_{n\to\infty} f(n)/g(n) = 1.$

1
INTRODUCTION

Preamble

This monograph deals with algorithms for a subclass of nonlinear, convex optimization problems, namely semidefinite programs, as well as selected applications of these problems. To place the topics which are dealt with in perspective, a short survey of the field of semidefinite programming[1] is presented in this chapter. Applications in combinatorial optimization and engineering are reviewed, after which interior point algorithms for this problem class are surveyed.

1.1 PROBLEM STATEMENT

Semidefinite programming (SDP) is a relatively new field of mathematical programming, and most papers on SDP were written in the 1990's, although its roots can be traced back a few decades further (see *e.g.* Bellman and Fan [19]). A paper on semidefinite programming from 1981 is descriptively named *Linear Programming with Matrix Variables* (Craven and Mond [37]), and this apt title may be the best way to introduce the problem.

The goal is to minimize the inner product

$$\langle C, X\rangle := \mathbf{Tr}\,(CX),$$

[1]Some authors prefer the more descriptive term 'optimization' to the historically entrenched 'programming'.

of two $n \times n$ symmetric matrices, namely a constant matrix C and a variable matrix X, subject to a set of constraints, where "Tr' denotes the trace (sum of diagonal elements) of a matrix.[2] The first of the constraints are linear:

$$\mathbf{Tr}\,(A_i X) = b_i, \quad i = 1, \ldots, m,$$

where the A_i's are given symmetric matrices, and the b_i's given scalars. Up to this point, the stated problem is merely a linear programming (LP) problem with the entries of X as variables. We now add the convex, nonlinear constraint that X must be symmetric positive semidefinite[3], denoted by $X \succeq 0$.[4]

The convexity follows from the convexity of the cone of positive semidefinite matrices. (We recommend that the reader briefly review the properties of positive semidefinite matrices in Appendix A.)

The problem under consideration is therefore

$$(P): \; p^* := \inf_X \left\{ \mathbf{Tr}\,(CX) \; : \mathbf{Tr}\,(A_i X) = b_i \; (i = 1, \ldots, m), \; X \succeq 0 \right\},$$

which has an associated Lagrangian dual problem:

$$(D): \; d^* := \sup_{y,S} \left\{ b^T y \; : \; \sum_{i=1}^m y_i A_i + S = C, \; S \succeq 0, \; y \in \mathbf{R}^m \right\}.$$

The duality theory of SDP is weaker than that of LP. One still has the familiar *weak duality* property: Feasible X, y, S satisfy

$$\mathbf{Tr}\,(CX) - b^T y = \mathbf{Tr}\left(\left(S + \sum_{i=1}^m y_i A_i \right) X \right) - \sum_{i=1}^m y_i \mathbf{Tr}\,(A_i X) = \mathbf{Tr}\,(SX) \geq 0,$$

where the inequality follows from $X \succeq 0$ and $S \succeq 0$ (see Theorem A.2 in Appendix A). In other words, the *duality gap* is nonnegative for feasible solutions.

Solutions (X, y, S) with zero duality gap

$$\mathbf{Tr}\,(CX) - b^T y = \mathbf{Tr}\,(SX) = 0$$

are optimal. For LP, if either the primal or the dual problem has an optimal solution, then both have optimal solutions, and the duality gap at optimality is zero. This is the

---

[2]This inner product corresponds to the familiar Euclidean inner product of two vectors – if the columns of the two matrices C and X are stacked to form vectors $\mathbf{vec}(X)$ and $\mathbf{vec}(C)$, then $\mathbf{vec}(C)^T \mathbf{vec}(X) = \mathbf{Tr}\,(CX)$. The inner product induces the so-called *Frobenius norm:*

$$\|A\|^2 := \langle A, A \rangle = \mathbf{Tr}\,\left(AA^T \right) = \sum_{i,j} A_{ij}^2.$$

See Appendix A for more details.

[3]By definition, a symmetric matrix X is positive semidefinite if $z^T X z \geq 0 \; \forall z \in \mathbf{R}^n$, or equivalently, if all eigenvalues of X are nonnegative.

[4]The symbol '$\succeq$' denotes the so-called Löwner partial order on the symmetric matrices: $A \succeq B$ means $A - B$ is positive semidefinite; see also Appendix A.

strong duality property. The SDP case is more subtle: One problem may be solvable and its dual infeasible, or the duality gap may be positive at optimality, *etc.* The existence of primal and dual optimal solutions is guaranteed if both *(P)* and (*D*) allow positive definite solutions, *i.e.* feasible $X \succ 0$ and $S \succ 0$. This is called the *Slater constraint qualification* (or Slater regularity condition). These duality issues will be discussed in detail in Chapter 2.

1.2 THE IMPORTANCE OF SEMIDEFINITE PROGRAMMING

SDP problems are of interest for a number of reasons, including:

- SDP contains important classes of problems as special cases, such as linear and quadratic programming (LP and QP);
- important applications exist in combinatorial optimization, approximation theory, system and control theory, and mechanical and electrical engineering;
- Loosely speaking, SDP problems can be solved to ϵ-optimality in polynomial time by *interior point algorithms* (see Section 1.9 for a more precise discussion of the computational complexity of SDP); interior point algorithms for SDP have been studied intensively in the 1990's, explaining the resurgence in research interest in SDP.

Each of these considerations will be discussed briefly in the remainder of this chapter.

1.3 SPECIAL CASES OF SEMIDEFINITE PROGRAMMING

If the matrix X is restricted to be diagonal, then the requirement $X \succeq 0$ reduces to the requirement that the diagonal elements of X must be nonnegative. In other words, we have an LP problem. Optimization problems with convex quadratic constraints are likewise special cases of SDP.[5] This follows from the well-known *Schur complement* theorem (Theorem A.9 in Appendix A). Thus we can represent the quadratic constraint

$$(Ax+b)^T(Ax+b) - (c^Tx+d) \leq 0, \quad x \in \mathbf{R}^n,$$

by the semidefmite constraint

$$\begin{bmatrix} I & Ax+b \\ (Ax+b)^T & c^Tx+d \end{bmatrix} \succeq 0.$$

In the same way, we can represent the *second order cone* (or 'ice cream cone'):

$$\left\{ (t,x) \;\middle|\; t \geq \sqrt{\sum_{i=1}^{n} x_i^2} \right\},$$

[5]This includes the well-known convex quadratic programming (QP) problem.

by requiring that a suitable arrow matrix be positive semidefinite:

$$\begin{bmatrix} tI & x \\ x^T & t \end{bmatrix} \succeq 0.$$

Another nonlinear example is

$$\min_x \left\{ \frac{(c^T x)^2}{d^T x} \;\middle|\; Ax \geq b \right\},$$

where it is known that $d^T x > 0$ if $Ax \geq b$. An equivalent SDP problem is:[6]

$$\min_{t,x} \left\{ t \;\middle|\; \begin{bmatrix} t & c^T x & 0 \\ c^T x & d^T x & 0 \\ 0 & 0 & \mathrm{Diag}\,(Ax - b) \end{bmatrix} \succeq 0 \right\}.$$

Several problems involving matrix norm or eigenvalue minimization may be stated as SDP's. A list of such problems may be found in Vandenberghe and Boyd [181]. A simple example is the classical problem of finding the largest eigenvalue $\lambda_{\max}(A)$ of a symmetric matrix *A*. The key observation here is that $t \geq \lambda_{\max}(A)$ if and only if $tI - A \succeq 0$. The SDP problem therefore becomes

$$\min_t \{t \mid tI - A \succeq 0, \quad t \in \mathbf{R}\}.$$

An SDP algorithm for this problem is described by Jansen *et al.* [88, 90].

1.4 APPLICATIONS IN COMBINATORIAL OPTIMIZATION

In this section we give a short review of some of the most important and successful applications of SDP in combinatorial optimization.

The Lovász ϑ-function

The most celebrated example of application of SDP to combinatorial optimization is probably the Lovász ϑ-function [115].

The Lovász ϑ-function maps a graph $G = (V, E)$ to $\mathbf{R}_+$, in such a way that

$$\omega(G) \leq \vartheta(\bar{G}) \leq \chi(G), \tag{1.1}$$

where $\omega(G)$ denotes the clique number[7] of G, $\chi(G)$ the chromatic number[8] of G, and $\bar{G}$ the complement of *G*.[9]

---

[6]We use the following notation: for a matrix *X*, diag *(X)* is the vector obtained by extracting the diagonal of *X;* for a vector x, $\mathrm{Diag}\,(x)$ is the diagonal matrix with the coordinates of *x* as diagonal elements.

[7]A maximum clique (or completely connected subgraph) is a subset $C \subset V$ with $\forall i,j \in C (i \neq j) : (i,j) \in E$, such that \C\ is as large as possible. The cardinality $|C|$ is called the clique number.

[8]The chromatic number is the number of colours needed to colour all vertices so that no two adjacent vertices share the same colour.

[9]The complementary graph (or complement) of $G = (V, E)$ is the graph $\bar{G} = (V, \bar{E})$ such that for each pair of vertices $i \neq j$ one has $(i,j) \in \bar{E}$ if and only if $(i,j) \notin E$.

The ϑ-function value is given as the optimal value of the following SDP problem:

$$\vartheta\left(\bar{G}\right) := \max_X \mathbf{Tr}\left(ee^T X\right) = e^T X e,$$

where $e \in \mathbf{R}^{|V|}$ denotes the vector of all-ones, subject to

$$\begin{aligned} x_{ij} &= 0,\ (i,j) \notin E\ (i \neq j) \\ \mathbf{Tr}\,(X) &= 1 \\ X &\succeq 0. \end{aligned}$$

The relation (1.1) is aptly known as the 'sandwich theorem' (a name coined by Knuth [103]). The sandwich theorem implies that $\vartheta(\bar{G})$ can be seen as a polynomial time approximation to both $\omega(G)$ and $\chi(G)$, and that the approximation cannot be off by more than a factor $|V|$. This may seem like a rather weak approximation guarantee, but some recent in-approximability results suggest that neither $\omega(G)$ nor $\chi(G)$ can be approximated within a factor $|V|^{1-\epsilon}$ for any $\epsilon > 0$ in polynomial time (see Håstad [81] and Feige and Kilian [57]).

We will review some of the properties of the ϑ-function in Chapter 10, and will give a proof of the sandwich theorem there; moreover, we will look at some alternative definitions of this function. We will also review how the ϑ-function may be used to estimate the Shannon capacity[10] of a graph.

The MAX—CUT problem and extensions

Another celebrated application of SDP to combinatorial optimization is the MAX-CUT problem. Consider a clique $G = (V, E)$ where each edge (i,j) has a given weight $w_{ij} \geq 0\ (i \neq j)$.

The goal is to colour all the vertices of G using two colours (say red and blue), in such a way that the total weight of edges where the endpoints have different colours (called *non—defect edges)* is as large as possible. The non—defect edges define a 'cut' in the graph — if one 'cuts' all the non—defect edges, then the blue and red vertices are separated. The total weight of the non—defect edges is therefore also called the weight of the cut.

Goemans and Williamson [66] derived a randomized 0.878-approximation algorithm[11] for this problem using SDP. The first step was to write the MAX-CUT problem as a Boolean quadratic optimization problem, *i.e.* an optimization problem with quadratic objective function and Boolean variables. For each vertex in V we introduce

[10]The Shannon capacity is a graph theoretical property that arises naturally in applications in coding theory; see Chapter 10 for details.

[11]Consider any class of NP-complete maximization problems. An α-approximation algorithm $(0 < \alpha \leq 1)$ for this problem class is then defined as follows. For any problem instance from this class, the algorithm terminates in time bounded by a polynomial in the size of the instance, and produces a feasible solution with objective value at least a times the optimal value for the instance.