LARGE HADRON COLLIDER PHENOMENOLOGY

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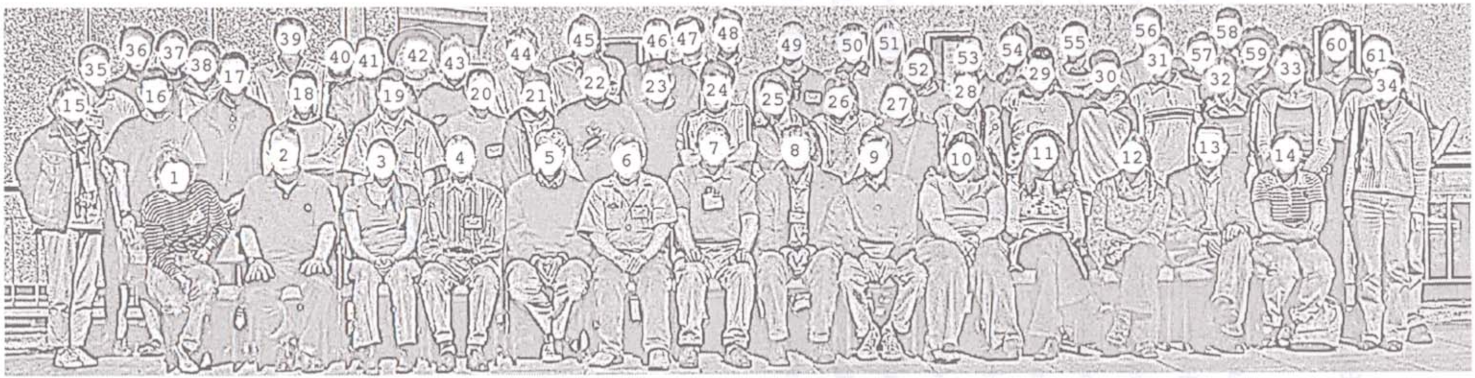

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Directors' Preface

With the Large Hadron Collider (LIIC) under construction, and due to come online in 2007, it was timely for a summer school in 2003 to focus oil LHC phenomenology. At a time when most of the experimental effort is directed to detector construction and software development, it was vitally important to focus young members of the experimental community on the physics that the LHC will deliver. At the same time, there is a continuing need to bring more young theorists into phenomenology, and in particular to inform them about the basic properties and capabilities of the machine. detectors and software. required for physics analysis. It was with this in mind that we set about organism,, the summer school. Senior postgraduate students and postdocs from all over the world were attracted to the lectures provided by the leading figures in their respective fields.

The lectures covered many aspects of LHC phenomenology and the research and development necessary to understand future data. Douglas Ross provided the foundation in his overview of the Standard Model with Keith Ellis emphasising the phenomenological development necessary to truly understand LEC data. John Ellis introduced the key concepts and ideas in going beyond the Standard Model and Andy Parker showed bow these are being translated into search methods for the general purpose detectors, Jim Virdee described the design and construction of these detectors and Rüdiger Schmidt gave a conceptual overview of the collider itself at a level that was appreciated by theorists and experimentalists alike, while Halls Hoffmann described the LHC computing challenge and how this was being met by Grid technology. The LHC provides a rich seam of physics: Val Gibson showed how current measurements would be extended on CP-violation in the b sector, Albert de Roeck presented the concepts necessary to understand forward and diffractive physics at the LHC, and Berndt Muller described how ALICE would explore the quark-gluon plasma when the collider runs with heavy ions.

We believe that this series of lectures will provide a thorough introduction to the phenomenology of LHC, not only for those currently working on this outstanding', endeavour, but also for those inspired by the breadth of physics covered but, perhaps, overawed by the scale and complexity of the detectors and the detailed understanding necessary to develop the phenomenology. While the wealth of data from the Tevatron will lead to new discoveries and, we hope, clues to the physics beyond the Standard Model, it is already clear that the model provides a very good description of the data, and a solid platform from which to analyse the processes that will underpin the physics at the LHC. These lectures will be a useful guide to these processes, and will enable the student, whether theorist or experimentalist, to judge the significance of these developments as they unfold.

Summer schools are not just about science: they are also about dialogue, discussion, meeting people and making friends. The discussion sessions in the evenings were spent probing the lecturers further, and each ended with a welcome pint. From the initial whisky tasting, sponsored by Chivas Regal, or the challenge of asking John Ellis an impossible question at the evening discussion sessions, or the guest appearance of Peter

Higgs at the school dinner, or the ne'er-to-be-forgotten sight of Keith Ellis in a kilt addressing the haggis, students, lecturers and organisers all appreciated the event. The school succeeded in its secondary aim, aided by a full social programme and a friendly environment provided by the staff of the John Burnet Hall. Within this, the scientific discussions and personal interactions flourished. Finally, the organising committee wish to acknowledge the guidance received from Ken Bowler, Peter Negus, David Saxon and Alan Walker at SUSSP and thank Linda Wilkinson, Emma Durrant and Leanne O'Donnell who all contributed to the school's overall success. We also wish to thank Colin Scott and Keith Geddes from Chivas Regal, who gave up their valuable time to provide a memorable whisky tasting, and to Ivan Reid, who became the unofficial school photographer.

The Organizing Committee acknowledge the support of the Scottish Universities Summer Schools in Physics, the Institute for Particle Physics Phenomenology at the University of Durham, and the Physics and Astronomy Departments of the Universities of Edinburgh, Glasgow and St. Andrews, without which the school would not have been possible.

We would also like to thank all of the lecturers and participants for their enthusiasm, both for physics and for life, which helped make this a truly memorable school.

Tony Doyle and James Stirling

Directors, October 2003

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Douglas Ross

University of Southampton, UK

1 Introduction

1.1 Quantum field theory

Particle physics is the study of particle interactions at the smallest possible scales. This immediately tells us that we need a quantum description of these particle interactions. In addition, we know from Heisenberg's uncertainty principle that in order to probe such short distances we need sufficiently high energies, so that most or all of the particles involved in a particular scattering event will be moving relativistically.

Quantum field theory is the consistent synthesis of quantum mechanics and special relativity applied to point particles. The goal of a particle theorist is to construct a quantum field theory, which accounts for the different interactions between particles and which can be used to make predictions for scattering cross-sections, decay rates and other measurable quantities, and it it is the task of particle experimentalists to perform the measurements in order to prove or disprove that a particular quantum field theory is consistent with experiment.

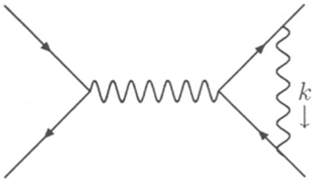
The techniques used to calculate measurable quantities from a quantum field theory are

(a) perturbation theory in which an expansion is performed in a power series of one or more coupling constants which are measures of the strength of interactions, and

(b) non-perturbative techniques such as lattice theories or numerical solutions of various equations derived from quantum field theory.

1.2 Renormalization

An additional difficulty that arises in quantum field theory but not in ordinary quantum mechanics is the occurrence of ultraviolet divergences. Higher order perturbative calculations involve summing over all possible intermediate states. In the language of Feynman diagrams this is represented by a loop and the Feynman rules require an integration over all possible momenta of particles inside the loop.



In general, this integral diverges for large momenta. Nowadays we do not interpret this divergence literally (as a genuine infinity) but assume that some new physics enters at some sufficiently high scale Λ to cutoff. The most successful new physics which does this job is string theory in which the point particle description is regarded as a low energy limit of a theory of extended objects (short strings). At 'low' energies the strings are in their ground state and can be described in terms of point-particle theories, but at some energy Λ, string excitations can be excited and in certain string theories these have been shown to conspire to cancel these ultraviolet divergences. From the 'low' energy viewpoint, the scale Λ of the string excitations acts as a cutoff for these divergences.

Nevertheless, these higher order perturbative corrections are very large since they depend (usually logarithmically) on this very large cutoff. This would render the perturbative expansion uncontrollable unless one can absorb the cutoff dependences by rescaling the quantum fields, masses, and coupling constants, which are the ingredients of the quantum field theory. This process of rescaling is called 'renormalization' and the goal is to rescale these parameters in such a way that in terms of rescaled parameters all physical quantities are finite (i.e. cutoff independent) in all perturbative orders.

Quantum field theories for which this is possible are called 'renormalizable theories'. The criteria for a theory to be renormalizable are:

(a) The interaction terms must not have dimension greater than four. Recalling that a bosonic field, ϕ, A_μ, … has dimension 1 whereas a fermiomc field, ψ, has dimension $\frac{3}{2}$, this means that interactions such as

$$\bar{\psi}\phi\psi, \quad \bar{\psi}\gamma^\mu\psi A_\mu, \quad \phi^4, \quad \text{and} \quad \phi^2 A^\mu A_\mu$$

are renormalizable, whereas interactions such as

$$\phi^6, \quad \bar{\psi}\gamma^\mu\gamma^\nu\psi\partial_\nu A_\mu, \quad \text{and} \quad \bar{\psi}\psi\bar{\psi}\psi$$

are not renormalizable.

(b) All propagators must vanish as the momentum of the propagating particle becomes infinite. This is usually the case, e.g. the propagator for a scalar particle of mass m and momentum p is

$$\frac{i}{(p^2 - m^2)}.$$

There is, however, one important exception, namely a massive vector (spin-one) particle, whose propagator is

$$-i\frac{\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{m^2}\right)}{(p^2 - m^2)},$$

which goes to a constant ~ $1/m^2$ as the momentum p becomes infinite. Note, however, that for a massless vector particle the propagator can be written as

$$-i\frac{g^{\mu\nu}}{p^2},$$

which *does* vanish in the ultraviolet limit as required.

1.3 Early quantum field theories

Until the 1970's the only successful renormalizable quantum field theory was QED – the quantum field theory describing the electromagnetic interactions. The interaction term between a charged fermion and a photon is

$$e\,\bar{\psi}\gamma^{\mu}\psi A_{\mu},$$

which has dimension four as required, and for which photon and fermion propagators both vanish as the momentum p tends to infinity. Furthermore, the coupling (charge) e is sufficiently small that a perturbative expansion in the fine structure constant

$$\alpha = \frac{e^2}{4\pi} = \frac{1}{137.06\ldots}$$

is reasonable.

Attempts to write quantum field theories in which nucleons interacted with pions via strong interactions through a (Yukawa) interaction term of the form

$$g\bar{\psi}_N\gamma^5\psi_N\phi_\pi$$

were unhelpful because the strong interactions require a large coupling constant, g. In fact a comparison of a leading order (tree-level) calculation of pion-nucleon scattering with measured cross-sections suggested a value of g such that

$$\frac{g^2}{4\pi} \approx 14,$$

rendering the perturbative expansion meaningless. At that time, no progress had been made on calculational techniques beyond perturbation theory such as lattice theories.

The weak interactions have small coupling, but were known to be described by an interaction which was the product of two currents. In terms of fermion fields the weak interaction is of the form

$$\mathcal{H}_{WK} \sim \bar{\psi}\gamma^{\mu}(1-\gamma^5)\psi\,\bar{\psi}\gamma_{\mu}(1-\gamma^5)\psi,$$

which is of dimension six and therefore non-renormalizable. Attempts were made to ameliorate this by introducing an 'intermediate vector boson', which was the progenitor of the $W^{\pm}$. In this theory the interaction term was of the form

$$\bar{\psi}\gamma^{\mu}(1-\gamma^5)\psi W_{\mu},$$

and the effective weak interaction Hamiltonian H_{WK} arose by the propagation of this vector boson between one current and the other. Now the interaction term was renormalizable, but it was known that the currents were vector and axial-vector so the intermediate boson had to be a vector particle and furthermore it had to be massive as the interactions are short-range. Thus once again the intermediate vector boson theory was not renormalizable.

1.4 Gauge theories

The feature of the successful theory QED that led to the development of the standard model, which is a renormalizable quantum field theory that describes strong, weak, and electromagnetic interactions, is the fact that it is an example of a 'gauge theory'.

A gauge theory is a theory which is invariant under a set of *local* transformations, i.e. transformations described by parameters that can vary in space-time. In the case of QED, the transformation is a phase transformation acting on the fields representing charged particles, but in general it can be a set of transformations (usually forming a group) such as isospin transformations on fields transforming as an isodoublet (or indeed any other isospin multiplet).

For each generator of the local transformations the invariance of the theory requires the introduction of a massless vector boson known as a 'gauge boson'. In the case of QED where there is only one transformation, there is one such massless vector particle which is identified as the photon. In the case of local isospin transformations one needs three such massless gauge bosons corresponding to the three possible isospin transformations.

1.5 A renormalizable theory of weak interactions

In weak interaction processes a particle undergoes an isospin transformation. For example, a neutron decaying into a proton and leptons can be viewed as an isospin transformation of a nucleon.

The extension of the idea of a gauge theory is therefore very appealing as a quantum field theory to describe weak interactions, in which the gauge bosons would be interpreted as the intermediate vector bosons.

However, gauge theories require that the gauge bosons be massless. This is suitable for electromagnetic interactions which are long-range (and possibly also gravitational interactions, which will not be discussed in these lectures), but unsuitable for weak interactions which are known to be short range and therefore involve the exchange of a massive particle.

We could simply break the gauge invariance explicitly by introducing a mass term for the gauge bosons. This would certainly not give a renormalizable theory and it would destroy the gauge invariance. There is a far more elegant way of breaking the gauge invariance known as 'spontaneous symmetry breaking'. In this scenario the invariance is maintained at the level of the Lagrangian, but the ground state of the system (the vacuum) is *not* invariant under the transformations. When spontaneous symmetry breaking is applied in a gauge theory, a mass is automatically generated for the gauge bosons. Such models were developed in the 1960's (independently) by Glashow, Weinberg, and Salam – but these authors did not address the problem of renormalization. Their model predicts the existence of a massive scalar particle – the elusive Higgs boson.

In 1971, Tini Veltman's graduate student, Gerard 't Hooft, showed that it was possible to exploit gauge invariance to cast the theory into a form in which the propagators of the massive particles did vanish as their momenta became infinite – thus fulfilling the condition for renormalizability which was absent in the intermediate vector boson theory. Thus we had a renormalizable quantum field theory for weak interactions.

1.6 A quantum field theory for strong interactions

In the 1960's Gell-Mann and Low, and independently Callan and Symanzik, showed that as a result of higher order perturbative corrections, a coupling 'constant' was not constant at all but that the effective coupling between particles varies with the energy at which the interaction takes place.

This variation is usually positive, i.e. the effective coupling increases with increasing energy. However, in 1973 Sidney Coleman's graduate student, David Politzer, showed that for non-Abelian gauge theories (a gauge theory in which the different transformations do not commute with each other – unlike the case of QED) the variation of the effective (or 'running') coupling *decreases* with energy.

This paved the way for the development of QCD – a gauge theory describing interactions involving quarks and gluons – as a model for the strong interactions. Although it would be the case that at the low energies at which experiments were performed in the 1960's the effective couplings are far too large for meaningful perturbative expansions, at the energy scales at which experiments are carried out today, the running coupling is sufficiently small that perturbative calculations can be carried out and the results compared with experiment.

2 QED as an Abelian gauge theory

Consider the Lagrangian density for a free Dirac field ψ:

$$\mathcal{L} = \overline{\psi}\,(i\gamma^\mu\partial_\mu - m)\,\psi. \tag{1}$$

Now this Lagrangian density is invariant under a phase transformation of the fermion field

$$\psi \rightarrow e^{i\omega}\psi,$$

since the conjugate field $\overline{\psi}$ transforms as

$$\overline{\psi} \rightarrow e^{-i\omega}\overline{\psi}.$$

The set of all such phase transformations is called the 'group U(1)' and it is said to be 'Abelian' which means that any two elements of the group commute. This just means that

$$e^{i\omega_1}e^{i\omega_2} = e^{i\omega_2}e^{i\omega_1}.$$

For the purposes of these lectures it will usually be sufficient to consider infinitesimal group transformations, i.e. we assume that the parameter ω is sufficiently small that we can expand in ω and neglect all but the linear term. Thus we write

$$e^{i\omega} = 1 + i\,\omega + \mathcal{O}(\omega^2).$$

Under such infinitesimal phase transformations the field ψ changes by $\delta\psi$ where

$$\delta\psi = i\,\omega\,\psi,$$

and the conjugate field $\overline{\psi}$ by $\delta\overline{\psi}$ where

$$\delta\overline{\psi} = -i\,\omega\,\overline{\psi},$$

such that the Lagrangian density remains unchanged (to order ω).

Now suppose that we wish to allow the parameter ω to depend on space-time. In that case, for infinitesimal transformations, we have

$$\delta\psi(x) = i\,\omega(x)\,\psi(x), \tag{2}$$

$$\delta\overline{\psi}(x) = -i\,\omega(x)\,\overline{\psi}(x). \tag{3}$$

Such local (i.e. space-time dependent) transformations are called 'gauge transformations'. Note now that the Lagrangian density (1) is *no longer* invariant under these transformations, because of the partial derivative that is interposed between $\overline{\psi}$ and ψ which will act on the space-time dependent parameter $\omega(x)$, such that the Lagrangian density changes by an amount $\delta\mathcal{L}$, where

$$\delta\mathcal{L} = -\overline{\psi}(x)\,\gamma^\mu\,(\partial_\mu\omega(x))\,\psi(x). \tag{4}$$

It turns out that we can repair the damage if we assume that the fermion field interacts with a vector field A_μ, called a 'gauge field', with an interaction term

$$-\,e\overline{\psi}\,\gamma^\mu A_\mu\psi$$

added to the Lagrangian density, which now becomes

$$\mathcal{L} = \overline{\psi}\,(i\gamma^\mu\,(\partial_\mu + i\,e\,A_\mu) - m)\,\psi. \tag{5}$$

In order for this to work we must also assume that apart from the fermion field transforming under a gauge transformation according to (2, 3), the gauge field, A_μ, also changes by δA_μ where

$$\delta A_\mu(x) = -\frac{1}{e}\partial_\mu\,\omega(x). \tag{6}$$

This change exactly cancels with eq.(4), so that once this interaction term has been added the gauge invariance is restored.

We recognize eq.(5) as being the fermionic part of the Lagrangian density for QED, where e is the electric charge of the fermion and A_μ is the photon field.

In order to have a proper quantum field theory, in which we can expand the photon field, A_μ, in terms of creation and annihilation operators for photons, we need a kinetic term for the field, A_μ, i.e. a term which is quadratic in the derivative of the field. We need to ensure that in introducing such a term we do not spoil the invariance under gauge transformations. This is achieved by defining the field strength, $F_{\mu\nu}$ as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{7}$$

It is easy to see that under the gauge transformation (6) each of the two terms on the right hand side of (7) changes, but the changes cancel out. Thus we may add to the Lagrangian any term which depends on $F_{\mu\nu}$ (and which is Lorentz invariant – so we must contract all Lorentz indices). Such a term is $a\, F_{\mu\nu}F^{\mu\nu}$ which gives the desired term which is quadratic in the derivative of the field A_μ. Furthermore, if we choose the constant a to be $-\frac{1}{4}$ then the Lagrange equations of motion match exactly the (relativistic formulation) of Maxwell's equations. (The determination of this constant a is the *only* place that a match to QED has been used. The rest of the Lagrangian density is obtained purely from the requirement of local U(1) invariance.)

We have thus arrived at the Lagrangian density for QED, but from the viewpoint of demanding invariance under U(1) gauge transformations rather than starting with Maxwell's equations and formulating the equivalent quantum field theory.

The Lagrangian density is:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}\left(i\gamma^\mu\left(\partial_\mu + i\,e\,A_\mu\right) - m\right)\psi. \tag{8}$$

Note that we are *not* allowed to add a mass term for the photon. A term such as $M^2 A_\mu A^\mu$ added to the Lagrangian density is not invariant under gauge transformations, but would give us a transformation

$$\delta\mathcal{L} = -\frac{2M^2}{e}A^\mu(x)\partial_\mu\omega(x).$$

Thus the masslessness of the photon can be understood in terms of the requirement that the Lagrangian be gauge invariant.

2.1 Covariant Derivatives

It is useful to introduce the concept of a 'covariant derivative'. This is not essential for Abelian gauge theories, but will be an invaluable tool when we extend these ideas to non-Abelian gauge theories.

The covariant derivative D_μ is defined to be

$$D_\mu = \partial_\mu + i\,e\,A_\mu. \tag{9}$$

This has the property that given the transformations of the fermion field (2) and the gauge field (6) the quantity

$$D_\mu\psi$$

is covariant under gauge transformations, i.e. it transforms by a phase rotation under gauge transformations.

We may thus rewrite the QED Lagrangian density as

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi. \tag{10}$$

Furthermore the field strength, $F_{\mu\nu}$, can be expressed in terms of the commutator of two covariant derivatives, i.e.

$$F_{\mu\nu} = -\frac{i}{e}\left[D_\mu, D_\nu\right] = -\frac{i}{e}[\partial_\mu, \partial_\nu] + [\partial_\mu, A_\nu] + [A_\mu, \partial_\nu] + i\,e\,[A_\mu, A_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{11}$$

2.2 Gauge fixing

There is a small difficulty that arises when we wish to quantize this theory. The Lagrange equation of motion for the photon field is

$$\left(-g^{\mu\nu}\partial^2 + \partial^\mu\partial^\nu\right) A_\nu = 0. \tag{12}$$

The propagator $G_{\mu\nu}(x-y)$ for the photon field is given by a solution to the Green function equation

$$\left(-g^{\mu\nu}\partial^2 + \partial^\mu\partial^\nu\right)_x G_{\nu\rho}(x-y) = i\delta^4(x-y)\delta^\mu_\rho. \tag{13}$$

Unfortunately this equation has no solution. What has gone wrong is that, because of the gauge invariance, a given value of the physical fields of the tensor $F_{\mu\nu}$ (the electric and magnetic fields) can be described by an infinite set of photon fields, all related by a gauge transformation.

It is therefore necessary to select a particular gauge in which to carry out the programme of quantization. This is done by imposing some constraint on the gauge field. There is an infinite range of possibilities, but the most convenient and most often used is the constraint or 'gauge condition'

$$\partial^\mu A_\mu = 0.$$

We impose this condition by adding the term

$$\frac{1}{(1-\xi)}\frac{1}{2}\left(\partial \cdot A\right)^2$$

to the Lagrangian density. Here ξ is a Lagrange multiplier called the 'gauge parameter'. After imposing this condition, the equation of motion, eq.(12), becomes

$$\left(-g^{\mu\nu}\partial^2 - \frac{\xi}{(1-\xi)}\partial^\mu\partial^\nu\right) A_\nu = 0, \tag{14}$$

and the Green function equation, eq.(13), changes to

$$\left(-g^{\mu\nu}\partial^2 - \frac{\xi}{(1-\xi)}\partial^\mu\partial^\nu\right)_x G_{\nu\rho}(x-y) = i\delta^4(x-y)\delta^\mu_\rho. \tag{15}$$

This has a simple solution whose Fourier transform gives the propagator of a photon with momentum p:

$$\tilde{G}^{\mu\nu}(p) = -i\frac{g^{\mu\nu} - \xi p^\mu p^\nu/k^2}{p^2}.$$

The most convenient choice of gauge parameter is $\xi = 0$. This is called the 'Feynman gauge' for which the photon propagator is simply

$$\tilde{G}^{\mu\nu}(p) = -i\frac{g^{\mu\nu}}{p^2}.$$