622 LECTURE NOTES IN ECONOMICS AND MATHEMATICAL SYSTEMS

Stefan Rostek

Option Pricing in Fractional Brownian Markets

Springer

Lecture Notes in Economics and Mathematical Systems 622

Stefan Rostek

Option Pricing in Fractional Brownian Markets

Springer

Dr. Stefan Rostek
University of Tübingen
Wirtschaftswissenschaftliches Seminar
Lehrstuhl für Betriebliche Finanzwirtschaft
Mohlstraße 36
72074 Tübingen
Germany
stefan.rostek@uni-tuebingen.de

ISSN 0075-8442
ISBN 978-3-642-00330-1 e-ISBN 978-3-642-00331-8
DOI 1 0.1007/978-3-642-00331-8
Springer Dordrecht Heidelberg London New York

Library of Congress Control Number: "PCN applied for"

Cover design: SPi Publishing Services

Printed on acid-free paper

springer is part & Springer Science+Business Media (www.springer.com)

To Ulrike

Foreword

Mandelbrot and van Ness (1968) suggested fractional Brownian motion as a parsimonious model for the dynamics of financial price data, which allows for dependence between returns over time. Starting with Rogers (1997) there is an ongoing dispute on the proper usage of fractional Brownian motion in option pricing theory. Problems arise because fractional Brownian motion is not a semimartingale and therefore "no arbitrage pricing" cannot be applied. While this is consensus, the consequences are not as clear. The orthodox interpretation is simply that fractional Brownian motion is an inadequate candidate for a price process. However, as shown by Cheridito (2003) any theoretical arbitrage opportunities disappear by assuming that market participants cannot react instantaneously.

This is the point of departure of Rostek's dissertation. He contributes to this research in several respects: (i) He delivers a thorough introduction to fractional integration calculus and uses the binomial approximation of fractional Brownian motion to give the reader a first idea of this special market setting. (ii) Similar to the classical work of Sethi and Lehoczky (1981) he compares Wick-Itô and Stratonovich integration for the unrestricted fractional Brownian case, obtaining deterministic option prices. This disproves in an elegant way several option pricing formulæ under fractional Brownian motion in the literature. (iii) If market prices move only slightly faster than any market participant can react, we are left with an incomplete market setting. Again, but now by a different reason, "no arbitrage pricing" cannot be applied. Based on Rostek and Schöbel (2006), he shows carefully and in great detail for the continuous as well as for the binomial setting that a risk preference based approach may be the solution to the option valuation puzzle under fractional Brownian motion.

I recommend this research monograph to everybody who is curious enough to learn more about the fragile character of our prevailing valuation theory.

Tübingen, December 2008 *Rainer Schöbel*

Acknowledgements

This book is the outcome of my three years lasting research work at the Department of Corporate Finance at the Eberhard Karls University of Tübingen. During this time I had the great fortune to be supported by a number of persons my heartfelt thanks go to. Moreover, I would like to single out the most important of these.

First and foremost, my thankfulness and appreciation are directed to my academic supervisor and teacher Prof. Dr.-Ing. Rainer Schöbel. His advice and guidance and particularly his scientific inquisitiveness accompanied by a perpetual positive mindset, heavily encouraged my work and formed the core of an ideal environment for my academic research. As a part of this stimulating environment, I would also like to thank Prof. Dr. Joachim Grammig for interesting discussions and advice, and not least for being the second referee of this thesis. Furthermore, my thanks go to my colleagues of the Corporate Finance Department Svenja Hager, Markus Bouziane, Robert Frontczak, Björn Lutz and Detlef Repplinger as well as Vera Klöckner for the friendly atmosphere and useful hints they provided.

I gratefully acknowledge the financial support of the Deutsche Forschungsgemeinschaft who funded my research as a member of the Research Training Group "Unternehmensentwicklung, Marktprozesse und Regulierung in dynamischen Entscheidungsmodellen".

I would like to express my deepest gratitude to my parents Roswitha and Franz Rostek. They were backing me all the way with their unrestricted faith in me and their enduring encouragement. Above all, I want to thank Ulrike Rostek, my beloved wife. Your patience, your understanding and your unconditional love are a godsend. Not knowing how to pay this off, I have to trust in Paul McCartney's 'fundamental theorem': "And in the end, the love you take is equal to the love you make." Thank you, you make everything so easy.

Schwieberdingen, December 2008 *Stefan Rostek*

Contents

1 Introduction 1

2 Fractional Integration Calculus 5
2.1 The Stochastic Process of Fractional Brownian Motion 7
2.2 Serial Correlation: The Role of the Hurst Parameter 13
2.3 The Wick-Based Approach to Fractional Integration 17
2.4 Pathwise and Stratonovich Integrals 22
2.5 Some Important Results of the Wick Type Fractional Integration Calculus 25
2.6 The S-Transform Approach 28

3 Fractional Binomial Trees 33
3.1 Binomial Approximation of an Arithmetic Fractional Brownian Motion Process 34
3.2 Binomial Approximation of the Conditional Moments of Fractional Brownian Motion 40
3.3 Binomial Approximation of a Geometric Fractional Price Process 44
3.4 Arbitrage in the Fractional Binomial Market Setting and Its Exclusion 49

4 Characteristics of the Fractional Brownian Market: Arbitrage and Its Exclusion 57
4.1 Arbitrage in the Unrestricted Continuous Time Setting 58
4.1.1 Arbitrage in the Continuous Setting Using Pathwise Integration 58
4.1.2 Arbitrage in the Continuous Time Setting Using Wick-Based Integration 60
4.2 Diverse Approaches to Exclude Arbitrage 62
4.2.1 Excluding Arbitrage by Extending the Wick Product on Financial Concepts 62

4.2.2 Regularization of Fractional Brownian Motion 67
4.2.3 Mixed Fractional Brownian Motion 67
4.2.4 Market Imperfections 68
4.3 On the Non-compatibility of Fractional Brownian Motion and Continuous Tradability 69
4.3.1 Itô and Stratonovich Formulations of the Classical Option Pricing Problem: The Work of Sethi and Lehoczky (1981) 69
4.3.2 Wick–Itô and Stratonovich Formulations of the Fractional Option Pricing Problem 73
4.4 Renouncement of Continuous Tradability, Exclusion of Arbitrage and Transition to Preference-Based Pricing 77

5 Risk Preference-Based Option Pricing in a Continuous Time Fractional Brownian Market 79
5.1 Motivation and Setup of the Model 79
5.2 The Conditional Distribution of Fractional Brownian Motion 81
5.2.1 Prediction Based on an Infinite Knowledge About the Past 81
5.2.2 Prediction Based on a Partial Knowledge About the Past 86
5.3 A Conditional Fractional Itô Theorem..................... 89
5.4 Fractional European Option Prices 92
5.5 The Influence of the Hurst Parameter 99
5.6 The Influence of Maturity and the Term Structure of Volatility 106

6 Risk Preference-Based Option Pricing in the Fractional Binomial Setting 111
6.1 The Two-Time Total Equilibrium Approach 113
6.2 The Two-Time Relative Equilibrium Approach 116
6.3 Multi-Time Equilibrium Approaches 121
6.3.1 Multi-Time Equilibria with Respect to Current Time t 121
6.3.2 Local Multi-Time Equilibria 125
6.4 Deeper Insights Provided by Discretization: The Continuous Time Case Reconsidered 128

7 Conclusion 131

References 135

Acronyms

σ	volatility parameter of the stock
μ	drift parameter of the stock
E	expectation operator
Var	variance operator
Cov	covariance operator
H	Hurst parameter
Γ	Gamma function
$\beta_{x,y}(z)$	incomplete Beta function
Ω	state space of random events
ω	random event or path
B_t^H	process of fractional Brownian motion at time t
t	current time
T	maturity time
$\mathbb{R}$	set of real numbers
B_t^H	process of Brownian motion at time t
τ	time to maturity
$\diamond$	Wick multiplier (diamond symbol)
$S(F)$	S-Transform of a function F
C_t	value of a European call option at time t
S_t	value of the basic risky asset at time t
$B_t^{H(n)}$	discrete n-step approximation of B_t^H
ξ	binomial random variable with zero mean and unit variance
n	number of discretization steps per unit of time
$\hat{B}_{T,t}^H$	conditional expectation of B_T^H at time t
$\hat{\sigma}_{T,t}^2$	conditional variance of B_T^H at time t
A_t	value of a deterministic bond
r	interest rate
K	strike price of a European option
S_0	initial price of the underlying of a European option
W_t^H	fractional White noise
$N(x)$	value of the standard normal distribution function

(I)	indicates Itô meaning of the following differential equation
(S)	indicates Stratonovich meaning of the following differential equation
(W)	indicates Wick-Itô meaning of the following differential equation
R_t	value of a dynamic portfolio at time t
P	probability measure on Ω
ρ_H	narrowing factor of the conditional distribution of fractional Brownian motion
fBm	fractional Brownian motion
$\hat{S}_t$	conditional stock price process
$\hat{B}_t$	conditional process of Brownian motion
$\bar{\mu}$	equilibrium drift rate
$\hat{B}_t^H$	conditional process of fractional Brownian motion
$\mathcal{F}_t$	information set available at time t
$I_{[\cdot,\cdot]}$	indicator function for a certain interval
η	partial derivative of the fractional call price with respect to the Hurst parameter H
ψ_0	digamma function

Chapter 1
Introduction

The vast majority of approaches towards option pricing deals with Brownian motion as a source of randomness. The seminal articles by Black and Scholes (1973) as well as by Merton (1973) crowned this evolution but did not conclude it by any means. Right up to today, the favorable properties and the well-developed stochastic calculus of classical Brownian motion attract both scientists and practitioners.

However, there was early evidence about some incompatibilities with regard to real market data. Concerning the stochastic process of Brownian motion, the main critique drawn from empiricism is at least two-fold:
On the one hand, real market distributions were shown to be not Gaussian (see e.g. Fama (1965)). The debate of recent years has put a great deal of effort on correcting this problem. Particularly the theory of Lévy processes allows it to incorporate a wide range of distributions into financial models. However, despite the large set of Lévy type stochastic processes, closed-form solutions are still limited to specific cases of non-Gaussian distributions. For more details about Lévy processes we refer the interested reader to the monograph of Cont and Tankov (2004) who provide a distinguished starting point to the topic.
On the other hand, the processes of observable market values seem to exhibit serial correlation (see e.g. Lo and MacKinley (1988)). Much less endeavor has been made to get a grip on this problem by factoring in aspects of persistence. However, at least there is one stochastic process that has often been proposed for mapping this kind of behavior: the very candidate is called fractional Brownian motion.

There are several reasons why we concern ourselves with this stochastic process. Fractional Brownian motion was originally introduced by Mandelbrot and van Ness (1968). It is a Gaussian stochastic process that is able to easily capture long-range dependencies or persistence. Being furthermore self-similar, its usage in financial models suggests itself. For reasons of parsimony,

S. Rostek, *Option Pricing in Fractional Brownian Markets.*
Lecture Notes in Economics and Mathematical Systems.

we appreciate that fractional Brownian motion possesses only one additional parameter, the so-called Hurst parameter, which lies between one and zero. Over the range of parameter values, the process shows different shapes of inter-temporal correlation. Particular interest arises from the fact that the case of serial independence is included. Therefore, fractional Brownian motion is an extension of classical Brownian motion. Comparing the respective results will both feed intuition and allow for a checking of plausibility.

The fundamental question of this thesis is whether and to what extent one can draw parallels between the fractional and the classical Brownian motion framework. More precisely: As fractional Brownian motion is an extension of Brownian motion, is it also possible to extend the respective theory of option pricing? Are the well-developed techniques of stochastic calculus transferable to fractional Brownian motion? Will we be faced with conceptual problems? Can we obtain closed-form solutions?

We will tackle all these problems step-by-step. Several times, we will switch over from discrete to continuous time considerations and vice versa. The reason for this is the following: Certainly, one could strictly separate the respective discussions and treat the cases one by one. However, so doing and starting with the continuous time case, we would miss the opportunity to motivate the results by those of the more descriptive discrete time setting. Turning the tables, if we discussed the discrete time framework first, we could not check the approximation results by comparing them with their limit case. By contrast, the alternating argumentation provides the best possible mutual benefit of the two frameworks, and additionally enhances the readability of the thesis.

In our preliminary Chap. 2, we will recall and present the most important insights concerning fractional Brownian motion and the corresponding integration calculus. We will become acquainted with the typical characteristics of the process. Concerning integration theory, we will get to know different concepts. In particular, it will be the so-called Wick-based integration calculus that will provide us with fractional analogues to the fundamental results of the well-known Itô calculus.
To get a first idea about the fractional Brownian market setting and the appendant characteristics, we will deal in Chap. 3 with a binomial approximation of fractional Brownian motion. For reasons of illustration, we will depict fractional binomial trees. These trees will not only enhance understanding of distributional aspects of fractional Brownian motion, they will also indicate the main problem of fractional Brownian markets: In an unrestricted market setting, arbitrage opportunities can occur.
In Chap. 4 we will readdress ourselves with the continuous time case. The problem of arbitrage will be thoroughly discussed. After presenting the scientific debate of the history, we will clarify that the problem can be solved

by restricting the set of feasible trading strategies. Motivated by the result from the discrete time framework, we will provide an elegant proof as to why a fractional Brownian market setting needs to be restricted. To this end, we will harness the reasoning of Sethi and Lehoczky (1981) and translate it into the fractional context. The result will be surprising at first glance but it will reveal perfectly the incompatibility of fractional Brownian motion and dynamical hedging. Consequently, we will renounce continuous tradability which is sufficient to ensure absence of arbitrage. As a proximate way out, we will suggest the transition to a risk preference based pricing approach.

Chapter 5 will form the core of this thesis and represents a further development of a preceding joint work by Rostek and Schöbel (2006). Assuming risk-neutral investors, we will price options in the continuous time fractional Brownian market. We will focus on a two-time valuation by postulating that the equilibrium condition we will introduce holds with respect to current time t and maturity T. We will apply some useful results concerning conditional expectation of fractional Brownian motion. Furthermore, we will state and use a conditional version of the fractional Itô theorem. Provided with these technical tools, we will be able to exploit the fundamental equilibrium condition. In the sense of a total equilibrium, the equilibrium condition will endogenously determine the drift of the underlying stock process. We will derive a closed-form solution for the price of a European option written on a stock that follows a fractional Brownian motion with arbitrary Hurst parameter H. Concerning the influence of the Hurst parameter H on the option price, we will elaborate different effects which we will call narrowing effect and maturity effect, respectively. Subsequently, we will consider the relation between option price and time to maturity which will lead us to the term structure of implied volatility. The latter will be a manifest result that clarifies the improvement our model yields.

By means of our derived results, we will be able to check how far appropriate results can also be drawn from our binomial approximation. In Chap. 6, we will therefore present the pricing approach from a discrete time vantage point. Like in the continuous time setting, we will first concentrate on a two-time valuation introducing a single equilibrium condition. We will address ourselves both to a relative and to an absolute equilibrium approach. Motivated by the ease and the traceability of the discrete time calculus, we will also consider multi-time equilibrium approaches. We will consider two different possibilities of stating the system of multi-time equilibrium conditions which will lead to totally different results. We will show that these results are in line with our understanding with respect to fractional Brownian motion.

We will finalize our dialectical consideration between discrete time and continuous time framework by making one further transition. In Sect. 6.4, we will use the deeper insight provided by the discrete multi-time results. In particular, we will ask ourselves what will happen if continuous time analogues of these multi-time equilibria are considered.

The concluding chapter summarizes the stated results and grants an outlook towards possible topics of further research.

Chapter 2
Fractional Integration Calculus

In order to model randomness in any stochastic model, one may do so by asserting a distribution of the random component. The somewhat more sophisticated approach—especially when modeling dynamical issues—is defining a suitable stochastic process. The overwhelming majority of treatable models based on stochastic processes deals with classical Brownian motion as the source of randomness. This is mainly due to the two main properties of this process, which are its Gaussian character, on the one hand, and its lack of serial correlation, on the other hand. However, though being easy to manage, these features often do not map things as they truly are. Real time series often fluctuate in a non-Gaussian fashion and/or are by all means serially correlated. A great deal of research effort has been invested to get a grip on the first problem; from the onset by introducing random jumps. Currently, researchers suggest so-called alpha-stable processes which are a special group of Levy processes. With the classical Brownian motion, these processes share the property of self-similarity.
However, in the literature of financial mathematics, few extensions have been proposed to overcome the assumption of independent increments for the stochastic processes. The most popular model was introduced by Mandelbrot and van Ness (1968). They hold true the Gaussian character of the process but allow for dependence over the line of time. Figure 2.1 by Cont and Tankov (2004) depicts the relations between important sets of stochastic processes. We see that while the intersection of all three sets is classical Brownian motion, fractional Brownian motion is still Gaussian and self-similar but no longer has independent increments.

From classical Brownian theory we learned that the transition from a deterministic framework to a stochastic one made it necessary to adjust the pertinent theory of integration. First of all, the definition of convergence had to be reconditioned in a mean square sense. Furthermore, the concept of a new integration calculus had to allow for the occurrence of infinite variation concerning the integrator. The celebrated solution to these problems was the

S. Rostek, *Option Pricing in Fractional Brownian Markets.*
Lecture Notes in Economics and Mathematical Systems.

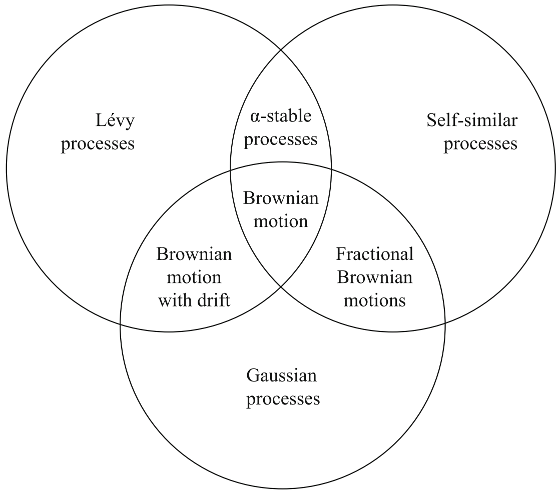

Fig. 2.1 Relations between different groups of stochastic processes (according to Cont/Tankov (2006))

approach to stochastic integration for semimartingales by Itô (1951), consequently named Itô calculus.

It can be shown (see Rogers (1997), p. 3–4) that—except the case $H = \frac{1}{2}$—fractional Brownian motion is not a semimartingale. This rules out the application of conventional integration theory. In other words, making the transition to the stochastic process of fractional Brownian motion, the Itô integration theory itself becomes obsolete.

Several suggestions have been made in the past to overcome these problems and to extend the integration concept of Itô to a more general concept. This chapter presents the most important ones. We start with the investigation of the so-called Wick-based approach due to Duncan et al. (2000). It can be regarded as a milestone towards an integration theory with respect to fractional Brownian motion as it demonstrates existing parallels to classical theory and facilitates the development of a fractional Brownian market setup. Comparing the Wick-based approach to the alternative concept of a fractional integral of Stratonovich type, these advantages will become clear. We will briefly recall the main results of the Wick calculus, including fractional versions of well-known theorems. These findings culminate in a fractional Itô theorem which was provided by Duncan et al. (2000).

Yet, the Wick-based approach in its original version by Duncan et al. (2000) is limited to the persistent cases with Hurst parameters larger than one half. The transition to the antipersistent case can only successfully be made by means of another still more general integration concept, the S-transform approach by Bender (for an overview, see Bender (2003a)). We sketch the basic idea of this approach. It can be viewed as seminal with respect to a clean mathematical foundation of fractional integration theory.

The outline of this chapter is as follows. In the first section, we define fractional Brownian motion and highlight important properties. We then investigate the role of the so-called Hurst parameter and see how persistence or serial correlation comes into play. The remaining sections of the chapter will demonstrate this in a technical way. We introduce approaches to a stochastic calculus for fractional Brownian motion and present important parallels to classical Brownian theory as a fractional Girsanov theorem or a fractional Itô theorem.

2.1 The Stochastic Process of Fractional Brownian Motion

We use the definition of fractional Brownian motion via its original presentation as a moving average of Brownian increments. We introduce the following defining notation for this purpose:

$$(x)_+^y = \begin{cases} x^y & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

For $0 < H < 1$, fractional Brownian motion $\left\{B_t^H, t \in \mathbb{R}\right\}$ is the stochastic process defined by

$$B_0^H(\omega) = 0 \quad \forall \omega \in \Omega, \tag{2.1}$$

$$B_t^H(\omega) = c_H \left[\int_{\mathbb{R}} \left((t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}\right) dB_s(\omega)\right], \tag{2.2}$$

where $\{B_s, s \in \mathbb{R}\}$ is a two-sided Brownian motion, H is the so-called Hurst parameter and

$$c_H = \sqrt{\frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma(\frac{1}{2}+H)\Gamma(2-2H)}}$$

is a normalizing constant. Note that for $t > 0$, B_t^H can be rewritten by

$$B_t^H = c_H \left[\int_{-\infty}^0 \left((t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}\right) dB_s + \int_0^t (t-s)^{H-\frac{1}{2}}\, dB_s\right].$$

Choosing the special parameter value $H = \frac{1}{2}$, we obtain

$$B_t^{\frac{1}{2}} = c_{\frac{1}{2}} \left[\int_{-\infty}^{0} \left((t-s)^{\frac{1}{2}-\frac{1}{2}} - (-s)^{\frac{1}{2}-\frac{1}{2}} \right) dB_s + \int_0^t (t-s)^{\frac{1}{2}-\frac{1}{2}} \, dB_s \right]$$
$$= \int_0^t dB_s = B_t,$$

where $$c_{\frac{1}{2}} = \sqrt{\frac{2 \cdot \frac{1}{2} \Gamma(\frac{3}{2} - \frac{1}{2})}{\Gamma(\frac{1}{2} + \frac{1}{2}) \Gamma(2 - 2 \cdot \frac{1}{2})}} = 1.$$

Obviously, $B_t^{\frac{1}{2}}$ coincides with classical Brownian motion. On the other hand, in the next section, the cases $0 < H < \frac{1}{2}$ and $\frac{1}{2} < H < 1$ will be identified with the occurrence of antipersistence and persistence, respectively. Consequently, fractional Brownian motion can be divided into three families exhibiting—as we will see—quite different properties.

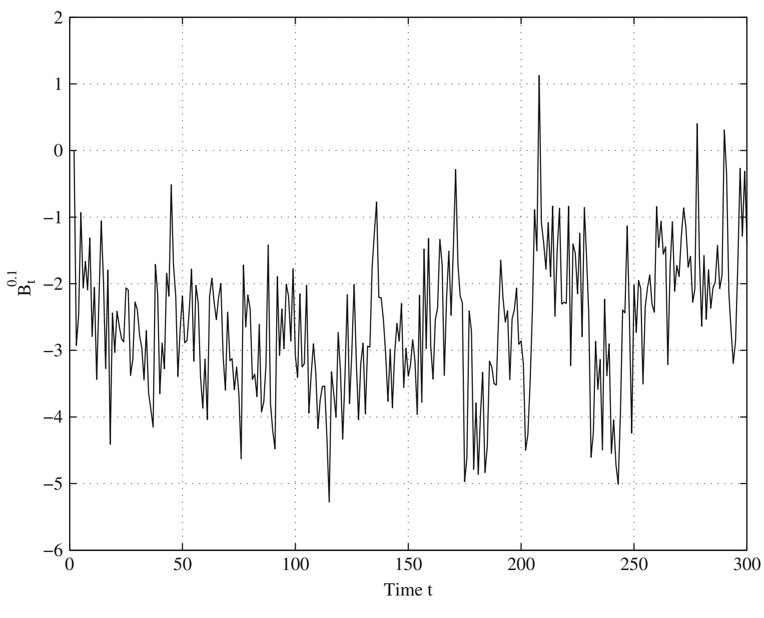

Fig. 2.2 Path of the fractional Brownian motion for $H = 0.1$

The Figs. 2.2–2.4 depict realizations of fractional Brownian motion. At first glance, we notice that the higher the Hurst parameter, the rougher the corresponding path. Looking on the scale of the axis, we also recognize that the smoother paths deviate considerably more from the zero mean. This is emphasized by Fig. 2.5 where the different processes are plotted in the same coordinate system. The next section will explain these phenomena in detail.

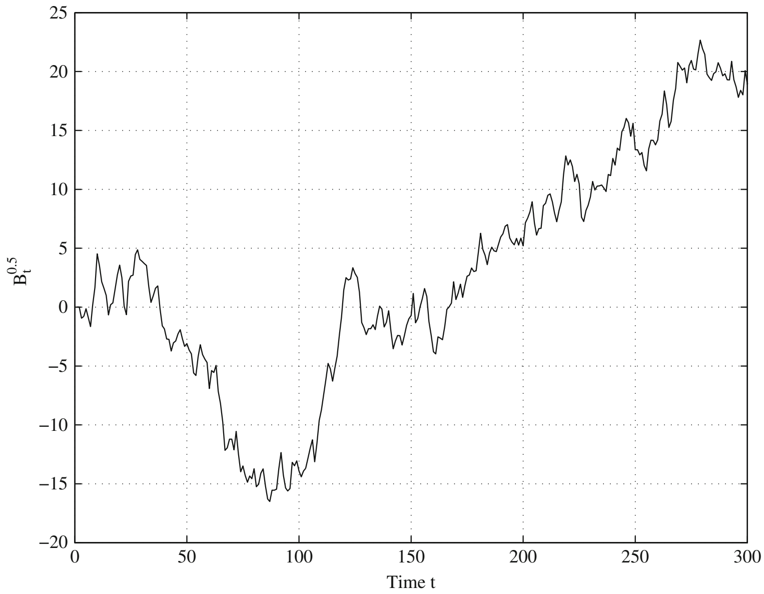

Fig. 2.3 Path of the fractional Brownian motion for $H = 0.5$

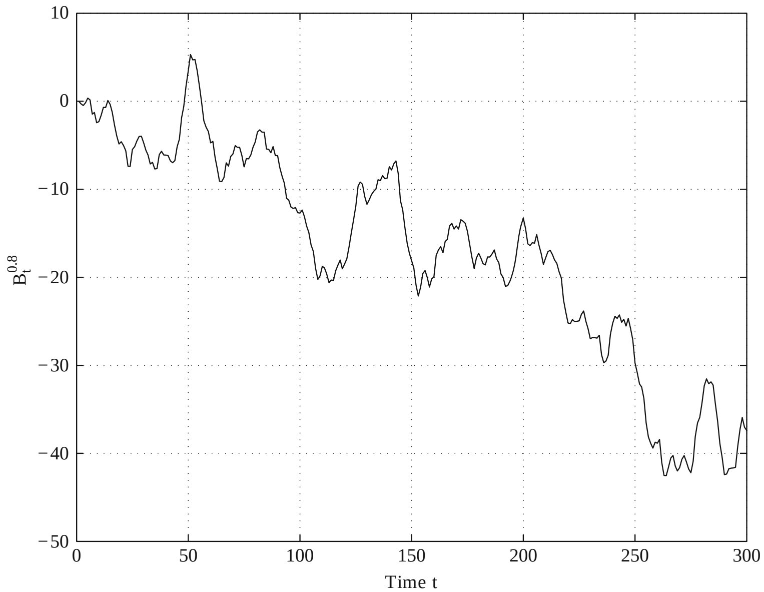

Fig. 2.4 Path of the fractional Brownian motion for $H = 0.8$