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Contents

Abstract

Consider a Hamiltonian action of a compact connected Lie group on a symplectic manifold (M, ω). Conjecturally, under suitable assumptions there exists a morphism of cohomological field theories from the equivariant Gromov-Witten theory of (M, ω) to the Gromov-Witten theory of the symplectic quotient. The morphism should be a deformation of the Kirwan map. The idea, due to D. A. Salamon, is to define such a deformation by counting gauge equivalence classes of symplectic vortices over the complex plane $\mathbb{C}$.

The present memoir is part of a project whose goal is to make this definition rigorous. Its main results deal with the symplectically aspherical case. The first one states that every sequence of equivalence classes of vortices over the plane has a subsequence that converges to a new type of genus zero stable map, provided that the energies of the vortices are uniformly bounded. Such a stable map consists of equivalence classes of vortices over the plane and holomorphic spheres in the symplectic quotient. The second main result is that the vertical differential of the vortex equations over the plane (at the level of gauge equivalence) is a Fredholm operator of a specified index.

Potentially the quantum Kirwan map can be used to compute the quantum cohomology of symplectic quotients.

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CHAPTER 1

Motivation and main results

1.1. Quantum deformations of the Kirwan map

Let (M, ω) be a symplectic manifold without boundary and G a compact connected Lie group with Lie algebra $\mathfrak{g}$. We fix a Hamiltonian action of G on M and an (equivariant) momentum map[1] $\mu : M \to \mathfrak{g}^*$. Throughout this memoir, we make the following standing assumption:

Hypothesis (H): *G acts freely on $\mu^{-1}(0)$ and the momentum map μ is proper.*

Then the symplectic quotient $\big(\overline{M} := \mu^{-1}(0)/G, \overline{\omega}\big)$ is well-defined, smooth and closed (i.e., compact and without boundary). The *Kirwan map* is a canonical ring homomorphism

$$\kappa : H_G^*(M) \to H^*(\overline{M}).$$

Here H^* and H_G^* denote cohomology and equivariant cohomology with rational coefficients, and the product structures are the cup products. F. Kirwan proved [**Kir**] that this map is surjective. Based on this result, the cohomology ring $H^*(\overline{M})$ was described in different ways by L. C. Jeffrey and F. Kirwan [**JK**, Theorem 8.1], S. Tolman and J. Weitsman [**TW**, Theorem 1], and many others.

The present memoir is concerned with the problem of "quantizing" the Kirwan map, which was first investigated by R. Gaio and D. A. Salamon. Assuming symplectic asphericity and some other restrictive conditions, in [**GS**, Corollary A'] these authors constructed a ring homomorphism from $H_G^*(M)$ to the (small) quantum cohomology of $(\overline{M}, \overline{\omega})$, which intertwines the Gromov-Witten invariants of the symplectic quotient with the symplectic vortex invariants. Their result is based on an adiabatic limit in the symplectic vortex equations. It was used by K. Cieliebak and D. A. Salamon in [**CS**, Theorem 1.3] to prove that given a monotone linear symplectic torus action on $\mathbb{R}^{2n}$ with minimal first equivariant Chern number at least 2, the quantum cohomology of $(\overline{M}, \overline{\omega})$ is isomorphic to the Batyrev ring.

The result by Gaio and Salamon motivates the following conjecture. We denote by $\mathrm{QH}_G^*(M, \omega)$ the equivariant quantum cohomology. By this we mean the $\mathbb{Q}$-vector space of all maps $\alpha : H_2^G(M, \mathbb{Z}) \to H_G^*(M)$ satisfying an equivariant version of the Novikov condition, together with a product counting holomorphic maps from S^2 to the fibers of the Borel construction for the action of G on M.[2] The Novikov condition states that for every number $C \in \mathbb{R}$ there are only finitely many classes $B \in H_2^G(M, \mathbb{Z})$, such that $\alpha(B) \neq 0$ and $\big\langle [\omega - \mu], B \big\rangle \leq C$. Here $[\omega - \mu] \in H_G^2(M)$ denotes the cohomology class of the two-cocycle $\omega - \mu$ in the Cartan model.

[1] Momentum maps are often called *moment maps* by symplectic geometers. However, the first term seems more appropriate, since the notion generalizes the linear and angular momenta appearing in classical mechanics.

[2] For the definition of this product see [**Gi**, **GiK**, **Kim**, **Lu**, **Ru**].

The space $\mathrm{QH}^*_G(M,\omega)$ is naturally a module over the equivariant Novikov ring Λ^μ_ω.[3] We denote by $\mathrm{QH}^*(\overline{M},\overline{\omega})$ the quantum cohomology of $(\overline{M},\overline{\omega})$ with coefficients in this ring. A map

$$\varphi : H_2^G(M,\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Q}}\big(H^*_G(M), H^*(\overline{M})\big) \tag{1.1}$$

satisfying the equivariant Novikov condition[4], induces a Λ^μ_ω-module homomorphism

$$\varphi_* : \mathrm{QH}^*_G(M,\omega) \to \mathrm{QH}^*(\overline{M},\overline{\omega}), \quad (\varphi_*\alpha)(B) := \sum \varphi(B_1)\alpha(B_2), \tag{1.2}$$

where the sum is over all pairs $B_1, B_2 \in H_2^G(M,\mathbb{Z})$ satisfying $B_1 + B_2 = B$. We denote by $c_1^G(M,\omega) \in H^2_G(M,\mathbb{Z})$ the first G-equivariant Chern class of (TM,ω), and by

$$N := \inf\big(\big\{\big\langle c_1^G(M,\omega), B\big\rangle \,\big|\, B \in H_2^G(M,\mathbb{Z}) : \text{ spherical}\big\} \cap \mathbb{N}\big) \in \mathbb{N} \cup \{\infty\}^5 \tag{1.3}$$

the minimal equivariant Chern number. We call (M,ω,μ) *semipositive* iff there exists a constant $c \in \mathbb{R}$ such that

$$\big\langle [\omega - \mu], B\big\rangle = c\big\langle c_1^G(M,\omega), B\big\rangle,$$

for every spherical class $B \in H_2^G(M,\mathbb{Z})$, and if $c < 0$ then $N \geq \frac{1}{2}\dim \overline{M}$.

1. Conjecture (Quantum Kirwan map, semipositive case). Assume that (H) holds and that (M,ω,μ) is convex at ∞ [6] and semipositive. Then there exists a map φ as in (1.1), satisfying the equivariant Novikov condition, such that the induced map φ_* as in (1.2) is a surjective ring homomorphism, and

$$\varphi(0) = \kappa, \tag{1.4}$$

$$\big\langle [\omega - \mu], B\big\rangle \leq 0,\ B \neq 0 \implies \varphi(B) = 0. \tag{1.5}$$

Once proven, this conjecture will give rise to a recursion formula for $\mathrm{QH}^*(\overline{M},\overline{\omega})$ in terms of $\mathrm{QH}^*_G(M,\omega)$ and φ.[7] As noticed in [**NWZ**], without the semipositivity condition, the conjecture likely needs to be modified as follows:

2. Conjecture (Quantum Kirwan map, general situation). Assume that (H) holds, and that (M,ω,μ) is convex at ∞. Then there exists a morphism of cohomological field theories (CohFT's) from the equivariant Gromov-Witten theory of (M,ω) to the Gromov-Witten theory of $(\overline{M},\overline{\omega})$.

For the notion of a morphism between two CohFT's V and W see [**NWZ**]. Such a morphism consists of a sequence of S_n-invariant multilinear maps

$$\psi^n : V^{\times n} \times H^*(\overline{M}_{n,1}(\mathbb{A})) \to W, \quad n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\},$$

[3]This ring consists of all maps $\lambda : H_2^G(M,\mathbb{Z}) \to \mathbb{Q}$ satisfying an equivariant version of the Novikov condition, analogous to the one above. The product is given by convolution.

[4]This condition is analogous to the one above.

[5]In this memoir $\mathbb{N} := \{1, 2, \ldots\}$ does not include 0.

[6]This means that there exists an ω-compatible and G-invariant almost complex structure J on M, such that the quadruple $\big(M,\omega,\mu,J\big)$ is convex at ∞ in the sense explained before Theorem 3 below.

[7]The recursion is over the set

$$\left\{\left\langle [\omega-\mu], \sum_{i=1}^k B_i\right\rangle \,\middle|\, k \in \mathbb{N}\cup\{0\},\ B_i \in H_2^G(M,\mathbb{Z}) : \varphi(B_i) \neq 0 \text{ or } \mathrm{GW}^G_{B_i} \neq 0,\ i = 1,\ldots,k\right\},$$

where $\mathrm{GW}^G_{B_i}$ denotes the 3-point genus 0 equivariant Gromov-Witten invariant of (M,ω) in the class B_i.

satisfying relations involving the composition maps of V and W. Here $\overline{M}_{n,1}(\mathbb{A})$ denotes the moduli space of stable n-marked scaled lines. Furthermore, the action of the symmetric group S_n is by permutations of the first n arguments of ψ^n. The map ψ^1 plays the role of φ_* as in Conjecture 1. The map ψ^0 measures how much ψ^1 fails to be a ring homomorphism. Once proven, Conjecture 2 will give rise to a recursion formula for the Gromov-Witten invariants of $(\overline{M},\overline{\omega})$ in terms of the equivariant Gromov-Witten invariants of (M,ω) and the morphism $(\psi^n)_n$.

The present memoir is part of a project whose goal is to prove Conjectures 1 and 2.[8] The approach pursued here was suggested by D. A. Salamon.[9] The idea is to construct the morphisms appearing in the conjectures by counting symplectic vortices over the complex plane $\mathbb{C}$. In a first step, we will consider the (symplectically) aspherical case. This means that

$$\int_{S^2} u^*\omega = 0, \quad \forall u \in C^\infty(S^2, M). \tag{1.6}$$

In this case the equivariant quantum cup product is induced by the ordinary cup product on $H^*_G(M)$.

1.2. Symplectic vortices, idea of the proof of existence of a quantum Kirwan map

To explain the idea of the proofs of Conjectures 1 and 2, we recall the symplectic vortex equations: Let J be an ω-compatible and G-invariant almost complex structure on M, $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ an invariant inner product on $\mathfrak{g}$, (Σ, j) a Riemann surface, and ω_Σ a compatible area form on Σ.[10] For every smooth (principal) G-bundle P over Σ we denote by $\mathcal{A}(P)$ the affine space of smooth connection one-forms on P, and by $C^\infty_G(P,M)$ the set of smooth equivariant maps from P to M. Consider the class

$$\widetilde{\mathcal{B}} := \widetilde{\mathcal{B}}_\Sigma := \big\{ w := (P,A,u) \,\big|\, P \text{ smooth } G\text{-bundle over } \Sigma, \; A \in \mathcal{A}(P),\, u \in C^\infty_G(P,M) \big\}. \tag{1.7}$$

The *symplectic vortex equations* are the equations

$$\bar\partial_{J,A}(u) = 0, \tag{1.8}$$

$$F_A + (\mu\circ u)\omega_\Sigma = 0 \tag{1.9}$$

for a triple $(P,A,u) \in \widetilde{\mathcal{B}}$. To explain these conditions, note that the pullback bundle $u^*TM \to P$ descends to a complex vector bundle $(u^*TM)/G \to \Sigma$.[11] For every $x \in M$ we denote by $L_x : \mathfrak{g} \to T_xM$ the infinitesimal action, corresponding to the action of G on M. With this notation, $\bar\partial_{J,A}(u)$ means the complex antilinear part of $d_Au := du + L_uA$, which we think of as a one-form on Σ with values in $(u^*TM)/G \to \Sigma$. In (1.9) we view the curvature F_A of A as a two-form on Σ with values in the adjoint bundle $\mathfrak{g}_P := (P\times\mathfrak{g})/G \to \Sigma$ [12]. Furthermore, identifying $\mathfrak{g}^*$ with $\mathfrak{g}$ via $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$, we view $\mu\circ u$ as a section of $\mathfrak{g}_P$. The vortex equations (1.8,1.9) were discovered by K. Cieliebak, A. R. Gaio, and D. A. Salamon [**CGS**], and

[8] Further relevant results will appear elsewhere, including [**Zi4**, **Zi5**].

[9] Private communication.

[10] This means that j and ω_Σ determine the same orientation of Σ.

[11] The complex structure on this bundle is induced by the almost complex structure J.

[12] Here G acts on $\mathfrak{g}$ in the adjoint way.

independently by I. Mundet i Riera [**Mu1**, **Mu2**].[13] A solution of these equations is called a *(symplectic) vortex*.

Two elements $w, w' \in \widetilde{\mathcal{B}}$ are called *(gauge) equivalent* iff there exists an isomorphism $\Phi : P' \to P$ of smooth G-bundles (which descends to the identity on Σ), such that

$$\Phi^*(A, u) := (A \circ d\Phi, u \circ \Phi) = (A', u').$$

In this case we write $w \sim w'$. We define

$$\mathcal{B} := \mathcal{B}_\Sigma := \widetilde{\mathcal{B}}/\sim . \tag{1.10}$$

The equations (1.8,1.9) are invariant under equivalence. We call an element $W \in \mathcal{B}$ a *vortex class* iff it consists of vortices. We define the *energy density* of a class $W \in \mathcal{B}$ to be

$$e_W := e_w := \frac{1}{2}\big(|d_A u|^2 + |F_A|^2 + |\mu \circ u|^2\big), \tag{1.11}$$

where $w := (P, A, u)$ is any representative of W. Here the norms are induced by the Riemannian metrics $\omega_\Sigma(\cdot, j\cdot)$ on Σ and $\omega(\cdot, J\cdot)$ on M, and by $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$. This definition does not depend on the choice of w. Vortex classes are absolute minimizers of the *(Yang-Mills-Higgs) energy functional*

$$E : \mathcal{B} \to [0, \infty], \quad E(W) := \int_\Sigma e_W \omega_\Sigma \tag{1.12}$$

in a given second equivariant homology class.[14] We define the *image* of a class $W \in \mathcal{B}$ to be the set of orbits of $u(P)$, where (P, A, u) is any representative of W. This is a subset of the orbit space M/G.

Consider now the complex plane $\Sigma := \mathbb{C}$, equipped with the standard area form $\omega_{\mathbb{C}} := \omega_0$.[15] Let $W \in \mathcal{B}_{\mathbb{C}}$ be a vortex class of finite energy, such that the image of W has compact closure[16]. Then W naturally carries an equivariant homology class $[W] \in H_2^G(M, \mathbb{Z})$. (See Section 3.1.) Let $B \in H_2^G(M, \mathbb{Z})$. We denote by $\mathcal{M}_B$ the set of vortex classes W representing the class B, and by $\mathrm{EG} \to \mathrm{BG}$ a universal G-bundle. There are natural evaluation maps

$$\mathrm{ev}_z : \mathcal{M}_B \to (M \times \mathrm{EG})/G, \quad \overline{\mathrm{ev}}_\infty : \mathcal{M}_B \to \overline{M}$$

at $z \in \mathbb{C}$ and $\infty \in \mathbb{C} \cup \{\infty\}$.[17] We denote by $\overline{\mathrm{PD}} : H_*(\overline{M}) \to H^*(\overline{M})$ the Poincaré duality map.

To prove Conjecture 1, heuristically, we define

$$\varphi : H_2^G(M, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Q}}\big(H_G^*(M), H^*(\overline{M})\big), \tag{1.13}$$

$$\big\langle \varphi(B)\alpha, \bar{b}\big\rangle := \int_{\mathcal{M}_B} \mathrm{ev}_0^*\,\alpha \smile \overline{\mathrm{ev}}_\infty^*\overline{\mathrm{PD}}(\bar{b}), \tag{1.14}$$

---

[13] In the case $G := S^1 \subseteq \mathbb{C}$ acting on $M := \mathbb{C}$ by multiplication, and $\Sigma := \mathbb{C}$, the corresponding energy functional was introduced previously by V. L. Ginzburg and L. D. Landau [**GL**], in order to model superconductivity. More generally, in the case $M := \mathbb{C}^n$ and G a closed subgroup of $\mathcal{U}(n)$, the functional appeared in physics in the context of *gauged linear sigma models*, starting with the work of E. Witten [**Wi**].

[14] See [**CGS**, Proposition 3.1]. Here we assume that Σ is closed, and vortices in the given homology class exist.

[15] In this case a vortex may be viewed as a map $(\Phi, \Psi, u) : \mathbb{C} \to \mathfrak{g} \times \mathfrak{g} \times M$, see Remark 6 below.

[16] with respect to the quotient topology on M/G

[17] See [**Zi1**, **Zi4**] and Section 2.1.

for $B \in H_2^G(M,\mathbb{Z})$, $\alpha \in H_G^*(M)$, and $\overline{b} \in H_*(\overline{M})$. Under the hypotheses of Conjecture 1, this map is "morally" well-defined and satisfies the conditions of the conjecture: If J is chosen as in the definition of convexity below, then there exists a compact subset of M/G containing the image of every finite energy vortex class $W \in \mathcal{B}_{\mathbb{C}}$ whose image has compact closure. This ensures that for every $B \in H_2^G(M,\mathbb{Z})$, the space $\mathcal{M}_B$ can be compactified by including holomorphic spheres in $\overline{M}$ and in the fibers of the Borel construction $(M \times \mathrm{EG})/G$. In the transverse case, it follows that the "boundary" of $\mathcal{M}_B$ has codimension at least 2. This makes the map φ "well-defined". It satisfies the equivariant Novikov condition as a consequence of the compactification argument, conservation of the equivariant homology class in the limit (see [**Zi1**, **Zi4**]), and the identity

$$E(W) = \big\langle [\omega - \mu], [W] \big\rangle.$$

This holds for every vortex class $W \in \mathcal{B}_{\mathbb{C}}$ of finite energy, such that the image of W has compact closure.[18] The identity also implies conditions (1.4,1.5).

The ring homomorphism property for the induced map $\mathrm{Q}\kappa := \varphi_*$ follows from an argument involving two marked points on the plane $\mathbb{C}$ that either move together or infinitely apart. The semipositivity assumption ensures that in the limit there is no bubbling of vortex classes over $\mathbb{C}$ without marked points. In contrast with holomorphic planes, such vortex classes may occur in stable maps in top dimensional strata, even in the transverse case. This is due to the fact that vortices over $\mathbb{C}$ "should not be rotated", which is explained below.

Surjectivity of $\mathrm{Q}\kappa$ will be a consequence of surjectivity of the Kirwan map κ, and the equivariant Novikov property.

The idea of the proof of Conjecture 2 is to define $\mathrm{Q}\kappa^1 := \mathrm{Q}\kappa$ as above, and for general $n \in \mathbb{N}_0$, $\mathrm{Q}\kappa^n$ in a similar way, using n marked points.

The "quantum Kirwan morphism" $(\mathrm{Q}\kappa^n)_{n\in\mathbb{N}_0}$ will intertwine the genus 0 symplectic vortex invariants with the Gromov-Witten invariants of $(\overline{M}, \overline{\omega})$. This will follow from a bubbling argument for a sequence of vortex classes over the sphere S^2, equipped with an area form that converges to ∞.[19]

The goal of the present memoir is to establish bubbling (i.e., "compactification") and Fredholm results for vortices over $\mathbb{C}$ in the aspherical case. Together with a transversality result (see [**Zi5**]), the Fredholm result will provide a natural structure of an oriented manifold on the set $\mathcal{M}_B$. Furthermore, the bubbling result will imply that the map

$$(\mathrm{ev}_0, \overline{\mathrm{ev}}_\infty) : \mathcal{M}_B \to (M \times \mathrm{EG})/G \times \overline{M}$$

is a pseudocycle[20].This will give a rigorous meaning to the integral (1.14). The ring homomorphism property and the relations defining a morphism of CohFT's will be a consequence of the bubbling result and a suitable gluing result.

[18] The equality follows from [**CGS**, Proposition 3.1] with $\Sigma := S^2 \cong \mathbb{C} \cup \{\infty\}$ and a smoothening argument at ∞.

[19] This corresponds to the adiabatic limit studied by Gaio and Salamon in [**GS**]. The new feature here is that in the limit, vortex classes over $\mathbb{C}$ may bubble off.

[20] as defined in [**MS2**, Definition 6.5.1]

1.3. Bubbling for vortices over the plane

To explain the first main result of this memoir, we assume that (M,ω) is aspherical, i.e., condition (1.6) is satisfied.[21] We denote by

$$\widetilde{\mathcal{M}} := \big\{(P,A,u)\in\widetilde{\mathcal{B}}_{\mathbb{C}}\,\big|\,(1.8,1.9)\big\},\quad \mathcal{M} := \widetilde{\mathcal{M}}/\sim \tag{1.15}$$

the class of all vortices over $\mathbb{C}$ and the set of equivalence classes of such vortices. The latter is equipped with a natural topology.[22] Consider the subspace of all classes in $\mathcal{M}$ with fixed finite energy $E>0$. There are three sources of non-compactness of this space: Consider a sequence $W_\nu\in\mathcal{M}$, $\nu\in\mathbb{N}$, of classes of energy E. In the limit $\nu\to\infty$, the following scenarios (and combinations) may occur:

Case 1. The energy density of W_ν blows up at some point in $\mathbb{C}$.

Case 2. There exists a number $r>0$ and a sequence of points $z_\nu\in\mathbb{C}$ that converges to ∞, such that the energy density of W_ν on the ball $B_r(z_\nu)$ is bounded above and below by some fixed positive constants.

Case 3. The energy densities converge to 0, i.e., the energy is spread out more and more.

In case 1, by rescaling W_ν around the bubbling point, in the limit $\nu\to\infty$, we obtain a non-constant J-holomorphic map from $\mathbb{C}$ to M. Using removal of singularity, this is excluded by the asphericity condition. In case 2, we pull W_ν back by the translation $z\mapsto z+z_\nu$, and in the limit $\nu\to\infty$, obtain a vortex class over $\mathbb{C}$. Finally, in case 3, we "zoom out" more and more. In the limit $\nu\to\infty$ and after removing the singularity at ∞, we obtain a pseudo-holomorphic map from S^2 to the symplectic quotient $\overline{M}=\mu^{-1}(0)/G$.

Hence in the limit, passing to a subsequence, we expect W_ν to converge to a new sort of stable map, which consists of vortex classes over $\mathbb{C}$ and pseudo-holomorphic spheres in $\overline{M}$. Here an important difference to Gromov-convergence for pseudo-holomorphic maps is the following: Although the vortex equations are invariant under all orientation preserving isometries of Σ, only translations on $\mathbb{C}$ should be allowed as reparametrizations used to obtain a vortex class over $\mathbb{C}$ in the limit. Hence we should disregard some symmetries of the equations. The reasons are that otherwise the reparametrization group does not always act with finite isotropy on the set of simple stable maps, and that there is no suitable evaluation map on the set of vortex classes, which is invariant under rotation.[23]

We are now ready to formulate the first main result. Here we say that (M,ω,μ,J) is *convex at* ∞ iff there exists a proper G-invariant function $f\in C^\infty(M,[0,\infty))$ and a constant $C\in[0,\infty)$, such that

$$\omega(\nabla_v\nabla f(x),Jv)-\omega(\nabla_{Jv}\nabla f(x),v)\geq 0,\quad df(x)JL_x\mu(x)\geq 0,$$

for every $x\in f^{-1}([C,\infty))$ and $0\neq v\in T_xM$. Here ∇ denotes the Levi-Civita connection of the metric $\omega(\cdot,J\cdot)$. This condition reduces to the existence of a

[21]The general situation is discussed in Remark 17 in Section 2.1.
[22]It is induced by the C^∞-topology on compact subsets of $\mathbb{C}$.
[23]See Remarks 22 and 33 in Sections 2.2 and 2.4.

plurisubharmonic function in the case in which G is trivial. It is satisfied e.g. if M is closed, and for linear actions on symplectic vector spaces.[24]

3. THEOREM (Bubbling). Assume that hypothesis (H) is satisfied, (M,ω) is aspherical, and (M,ω,μ,J) is convex at ∞. Let $k \in \mathbb{N}_0$, and for $\nu \in \mathbb{N}$ let $W_\nu \in \mathcal{M}$ be a vortex class and $z_1^\nu, \dots, z_k^\nu \in \mathbb{C}$ be points. Suppose that the closure of the image of each W_ν is compact, and

$$E(W_\nu) > 0, \forall \nu \in \mathbb{N}, \quad \sup_{\nu \in \mathbb{N}} E(W_\nu) < \infty.$$

Then there exists a subsequence of $\big(W_\nu, z_0^\nu := \infty, z_1^\nu, \dots, z_k^\nu\big)$ that converges to some genus 0 stable map $(\mathbf{W}, \mathbf{z})$ consisting of vortex classes over $\mathbb{C}$ and pseudo-holomorphic spheres in $\overline{M}$, with $k+1$ marked points. (See Definitions 15 and 20 in Chapter 2.[25])

The proof of this result combines Gromov compactness for pseudo-holomorphic maps with Uhlenbeck compactness. It relies on work [**CGMS**, **GS**] by K. Cieliebak, R. Gaio, I. Mundet i Riera, and D. A. Salamon. The idea is the following. In order to capture all the energy, we "zoom out rapidly", i.e., rescale the vortices so much that the energies of the rescaled vortices are concentrated near the origin in $\mathbb{C}$. Now we "zoom back in" in such a way that we capture the first bubble, which may either be a vortex class over $\mathbb{C}$ or a J-holomorphic sphere in $\overline{M}$. In the first case we are done. In the second case we "zoom in" further, to obtain a finite number of vortices and spheres that are attached to the first bubble. Iterating this procedure, we construct the limit stable map.

The proof involves generalizations of results for pseudo-holomorphic maps to vortices: a bound on the energy density of a vortex, quantization of energy, compactness with bounded derivatives, and hard and soft rescaling. The proof that the bubbles connect and no energy is lost between them, uses an isoperimetric inequality for the invariant symplectic action functional, proved in [**Zi2**], based on a version of the inequality by R. Gaio and D. A. Salamon [**GS**].

Another crucial point is that when "zooming out", no energy is lost locally in $\mathbb{C}$ in the limit. This relies on an upper bound on the "momentum map component" of a vortex, due to R. Gaio and D. A. Salamon.

1.4. Fredholm theory for vortices over the plane

The space of gauge equivalence classes of symplectic vortices can be viewed as the zero set of a section of an infinite dimensional vector bundle. Formally, the second main result of this memoir states that in the case $\Sigma = \mathbb{C}$ the vertical differential of this section is Fredholm when seen as an operator between suitable weighted Sobolev spaces. We will first state the result and then interpret it in this way.

Statement of the Fredholm result. Consider the case $\Sigma := \mathbb{C}$ and $\omega_{\mathbb{C}} := \omega_0$. Let $p > 2$ and λ be real numbers.[26] We define the set $\mathcal{B}^p_\lambda$ as follows. For a

[24]See [**CGMS**, Example 2.8]. Here the standing assumption that μ is proper is used.

[25]The reasons for introducing the additional marked points $z_0^\nu = \infty$ are explained in Remark 21 in Section 2.2.

[26]In this memoir, p and λ always refer to finite values, unless otherwise stated.

measurable function $f:\mathbb{R}^n\to\mathbb{R}$ we denote $\|f\|_p:=\|f\|_{L^p(\mathbb{C})}\in[0,\infty]$ and define the λ*-weighted* L^p*-norm of* f to be

$$\|f\|_{p,\lambda}:=\big\|(1+|\cdot|^2)^{\frac{\lambda}{2}}f\big\|_p\in[0,\infty].$$

We define $\widetilde{\mathcal{B}}^p_{\mathrm{loc}}$ to be the class consisting of all triples (P,A,u), where $P\to\mathbb{C}$ is a G-bundle of class $W^{2,p}_{\mathrm{loc}}$ [27], A a connection (one-form) of class $W^{1,p}_{\mathrm{loc}}$, and $u:P\to M$ a G-equivariant map of class $W^{1,p}_{\mathrm{loc}}$. We call two elements $w,w'\in\widetilde{\mathcal{B}}^p_{\mathrm{loc}}$ *p-equivalent* iff there exists an isomorphism $\Phi:P'\to P$ of G-bundles of class $W^{2,p}_{\mathrm{loc}}$ (descending to the identity on $\mathbb{C}$), such that $\Phi^*(A,u)=(A',u')$. In this case we write $w\sim_p w'$. We define

$$\widetilde{\mathcal{B}}^p_\lambda:=\big\{w:=(P,A,u)\in\widetilde{\mathcal{B}}^p_{\mathrm{loc}}\,\big|\,\overline{u(P)}\text{ compact},\ \|\sqrt{e_w}\|_{p,\lambda}<\infty\big\},\tag{1.16}$$

$$\mathcal{B}^p_\lambda:=\widetilde{\mathcal{B}}^p_\lambda/\sim_p,\tag{1.17}$$

where the energy density e_w is defined as in (1.11).

Let $W\in\mathcal{B}^p_\lambda$. We define normed vector spaces $\mathcal{X}^{p,\lambda}_W$ and $\mathcal{Y}^{p,\lambda}_W$ as follows. Let E be a real vector bundle over $\mathbb{C}$. We denote by $A^i(E)$ the bundle of alternating i-forms on $\mathbb{C}$ with values in E. If E is a complex vector bundle, then we denote by $A^{0,1}(E)$ the bundle of anti-linear one-forms on $\mathbb{C}$ with values in E. We denote by $\Gamma^p_{\mathrm{loc}}(E)$ and $\Gamma^{1,p}_{\mathrm{loc}}(E)$ the spaces of its L^p_{loc}- and $W^{1,p}_{\mathrm{loc}}$-sections, respectively. We fix $w:=(P,A,u)\in\widetilde{\mathcal{B}}^p_\lambda$, and denote by

$$\mathfrak{g}_P:=(P\times\mathfrak{g})/G\to\Sigma,\quad TM^u:=(u^*TM)/G\to\mathbb{C}$$

the adjoint bundle and the quotient of the pullback bundle $u^*TM\to P$. Let $\zeta:=(\alpha,v)\in\Gamma^{1,p}_{\mathrm{loc}}\big(A^1(\mathfrak{g}_P)\oplus TM^u\big)$. We denote $d_A\alpha:=d\alpha+[A\wedge\alpha]$ and by ∇^A the connection on TM^u induced by the Levi-Civita connection ∇ of $\omega(\cdot,J\cdot)$ and A [28], and we abbreviate $\nabla^A\zeta:=(d_A\alpha,\nabla^Av)$. We define the weighted norm

$$\|\zeta\|_{w,p,\lambda}:=\|\zeta\|_\infty+\big\||\nabla^A\zeta|+|d\mu(u)v|+|\alpha|\big\|_{p,\lambda}\in[0,\infty].\tag{1.18}$$

Here the norms are taken with respect to $\omega(\cdot,J\cdot)$, the standard metric on $\mathbb{C}$, and $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$. We denote by $*$ the Hodge star operator with respect to the standard metric on $\mathbb{C}$, and by

$$d_A^*=-*d_A*:\Gamma^{1,p}_{\mathrm{loc}}(A^1(\mathfrak{g}_P))\to\Gamma^p_{\mathrm{loc}}(\mathfrak{g}_P)$$

the formal adjoint of the twisted differential d_A. For $x\in M$ we denote by $L_x^*:T_xM\to\mathfrak{g}$ the adjoint map of the infinitesimal action of G on M at x, with respect to the Riemannian metric $\omega(\cdot,J\cdot)$ and the inner product $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$. The collection $(L_x^*)_{x\in M}$ induces a map L_u^* from the space of sections of TM^u to the space of sections of $\mathfrak{g}_P$. We define[29]

$$L_w^*:\Gamma^{1,p}_{\mathrm{loc}}\big(A^1(\mathfrak{g}_P)\oplus TM^u\big)\to\Gamma^p_{\mathrm{loc}}(\mathfrak{g}_P),\quad L_w^*(\alpha,v):=-d_A^*\alpha+L_u^*v,\tag{1.19}$$

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[27] By definition, P is a topological G-bundle over $\mathbb{C}$, equipped with an atlas of local trivializations whose transition functions lie in $W^{2,p}_{\mathrm{loc}}$. Every such bundle is trivializable, but we do not fix a trivialization here.

[28] See definition (A.63) in Appendix A.7.

[29] As explained in the next subsection, the map L_w^* is the formal adjoint for the infinitesimal action of the gauge group on the product of the spaces of connections and equivariant maps from P to M.

$$\widetilde{\mathcal{X}}_w^{p,\lambda} := \big\{\zeta \in \Gamma^{1,p}_{\mathrm{loc}}\big(A^1(\mathfrak{g}_P) \oplus TM^u\big) \,\big|\, L_w^*\zeta = 0,\ \|\zeta\|_{w,p,\lambda} < \infty\big\}, \tag{1.20}$$

$$\widetilde{\mathcal{Y}}_w^{p,\lambda} := \big\{\zeta' \in \Gamma^{p}_{\mathrm{loc}}\big(A^{0,1}(TM^u) \oplus A^2(\mathfrak{g}_P)\big) \,\big|\, \|\zeta'\|_{p,\lambda} < \infty\big\}, \tag{1.21}$$

$$\mathcal{X}_W^{p,\lambda} := \coprod_{w\in W} \widetilde{\mathcal{X}}_w^{p,\lambda}/\sim_p, \tag{1.22}$$

$$\mathcal{Y}_W^{p,\lambda} := \coprod_{w\in W} \widetilde{\mathcal{Y}}_w^{p,\lambda}/\sim_p\ . \tag{1.23}$$

Here the equivalence relations in (1.22,1.23) are defined similarly to the p-equivalence relation on $\widetilde{\mathcal{B}}^{p,\lambda}$. Since the energy-density is invariant under the gauge transformations, the gauge group of P [30] of class $W^{2,p}_{\mathrm{loc}}$ naturally acts on the set

$$\widetilde{\mathcal{B}}^p_\lambda(P) := \big\{(A,u) \,\big|\, (P,A,u) \in \widetilde{\mathcal{B}}^p_\lambda\big\}. \tag{1.24}$$

Assume that $\lambda > 1 - 2/p$. Then this action is free.[31] Therefore, $\mathcal{X}_W^{p,\lambda}$ is naturally a normed vector space, which can canonically be identified with $\widetilde{\mathcal{X}}_w^{p,\lambda}$, for any representative w of W. Similarly, $\mathcal{Y}_W^{p,\lambda}$ is naturally a normed vector space, which may be identified with $\widetilde{\mathcal{Y}}_w^{p,\lambda}$, for any representative w of W.

Consider the operator

$$\mathcal{D}_W^{p,\lambda} : \mathcal{X}_W^{p,\lambda} \to \mathcal{Y}_W^{p,\lambda} \tag{1.25}$$

$$\mathcal{D}_W^{p,\lambda}[w;\alpha,v] := \begin{bmatrix} \big(\nabla^A v + L_u\alpha\big)^{0,1} - \frac{1}{2}J(\nabla_v J)(d_A u)^{1,0} \\ d_A\alpha + \omega_0\, d\mu(u)v \end{bmatrix}. \tag{1.26}$$

Here the brackets $[\cdots]$ denote equivalence classes. Formally, this is the vertical differential of a section of a Banach space bundle over a Banach manifold, whose zeros are the gauge equivalence classes of vortices. (For explanations see the next subsection.) Recall that $c_1^G(M,\omega) \in H_2^G(M,\mathbb{Z})$ denotes the equivariant first Chern class of (M,ω). The second main result of this memoir is the following.

4. Theorem (Fredholm property). Let $p > 2$, $\lambda \in \mathbb{R}$, and $W \in \mathcal{B}^p_\lambda$ (defined as in (1.17)). Assume that $\dim M > 2\dim G$. Then the following statements hold.

(i) If $\lambda > 1 - 2/p$ then the normed vector spaces $\mathcal{X}_W^{p,\lambda}$ and $\mathcal{Y}_W^{p,\lambda}$ (defined as in (1.22,1.23)) are complete.

(ii) If $1 - 2/p < \lambda < 2 - 2/p$ then the operator $\mathcal{D}_W^{p,\lambda}$ (defined as in (1.25,1.26)) is well-defined and Fredholm of real index

$$\mathrm{ind}\mathcal{D}_W^{p,\lambda} = \dim M - 2\dim G + 2\big\langle c_1^G(M,\omega), [W]\big\rangle, \tag{1.27}$$

where $[W]$ denotes the equivariant homology class of W (see Section 3.1).

The contraction appearing in formula (1.27) can be interpreted as a certain Maslov index. (See Proposition 62 in Section 3.1.) The condition $1 - 2/p < \lambda < 2 - 2/p$ in part (ii) of this result captures the geometry of finite energy vortices over $\mathbb{C}$. (See Remark 9 below.) The condition $\lambda < 2 - 2/p$ is also needed for the map $\mathcal{D}_W^{p,\lambda}$ to have the right Fredholm index. (See Remark 10.) The definition of the space $\mathcal{X}_W^{p,\lambda}$ naturally parallels the definition of $\mathcal{B}^p_\lambda$. (See Remark 11.) Note that some naive choices for the domain and target of the operator $\mathcal{D}_W^{p,\lambda}$ do not work. (See Remark 12.)

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[30] i.e., the group of transformations on P

[31] See Lemma 121 in Appendix A.7.

The proof of Theorem 4 is based on a Fredholm result for the augmented vertical differential and the existence of a bounded right inverse for L_w^*. (See Theorems 63 and 64 in Section 3.2.1.) The proof of Theorem 63 has two main ingredients. The first one is the existence of a suitable complex trivialization of the bundle $A^1(\mathfrak{g}_P)\oplus TM^u$. For R large, $z\in\mathbb{C}\setminus B_R$ and $p\in\pi^{-1}(z)\subseteq P$ such a trivialization respects the splitting

$$T_{u(p)}M=(\mathrm{im}L^{\mathbb{C}}_{u(p)})^\perp\oplus\mathrm{im}L^{\mathbb{C}}_{u(p)}, \tag{1.28}$$

where $L_x^{\mathbb{C}}:\mathfrak{g}\otimes\mathbb{C}\to T_xM$ denotes the complexified infinitesimal action, for $x\in M$. The second ingredient are two propositions stating that the standard Cauchy-Riemann operator $\partial_{\bar z}$ and a related matrix differential operator are Fredholm maps between suitable weighted Sobolev spaces. These results are based on the analysis of weighted Sobolev spaces carried out by R. B. Lockhart and R. C. McOwen [**Lo1**, **Lo2**, **Lo3**, **LM1**, **LM2**, **McO1**, **McO2**, **McO3**]. A crucial analytical ingredient is a Hardy-type inequality (Proposition 91 in Appendix A.4).

Motivation, a formal setting. To put the Fredholm result into context, let (Σ,j) be a connected smooth Riemann surface, equipped with a compatible area form ω_Σ. Recall the definitions (1.7,1.10) of $\widetilde{\mathcal{B}},\mathcal{B}$. Consider the subclass $\widetilde{\mathcal{B}}^*\subseteq\widetilde{\mathcal{B}}$ of triples (P,A,u) for which there exists a point $p\in P$, such that the action of G at the point $u(p)\in M$ is free. We define

$$\mathcal{B}^*:=\widetilde{\mathcal{B}}^*/\sim,$$

where $\sim$ is defined as before the definition (1.10). Formally, $\mathcal{B}^*$ may be viewed as an infinite dimensional manifold, since for every smooth G-bundle P over Σ, the natural action of the gauge group $\mathcal{G}_P=C^\infty_G(P,G)$ on the "infinite dimensional manifold"

$$\widetilde{\mathcal{B}}^*_P:=\big\{(A,u)\,\big|\,(P,A,u)\in\widetilde{\mathcal{B}}^*\big\} \tag{1.29}$$

is free. Furthermore, the set of vortex classes may be viewed as the zero set of a section of a vector bundle $\mathcal{E}$ over $\mathcal{B}^*$, with infinite dimensional fiber, as follows. Consider the "vector bundle" $\widetilde{\mathcal{E}}:=\widetilde{\mathcal{E}}_\Sigma$ over $\widetilde{\mathcal{B}}^*$, whose fiber over a point $w=(P,A,u)\in\widetilde{\mathcal{B}}^*$ is given by

$$\widetilde{\mathcal{E}}_w:=\Gamma\left(A^{0,1}(TM^u)\oplus A^2(\mathfrak{g}_P)\right).^{32} \tag{1.30}$$

The bundle $\mathcal{E}:=\mathcal{E}_\Sigma$ over $\mathcal{B}^*$ is now defined to be the quotient of the bundle $\widetilde{\mathcal{E}}\to\widetilde{\mathcal{B}}^*$ under the natural equivalence relation lifting the relation $\sim$ on $\widetilde{\mathcal{B}}^*$. Finally,

$$\mathcal{S}:\mathcal{B}^*\to\mathcal{E}$$

is defined to be the section induced by

$$\widetilde{\mathcal{S}}:\widetilde{\mathcal{B}}^*\to\widetilde{\mathcal{E}},\quad \widetilde{\mathcal{S}}(A,u):=\big(\bar\partial_{J,A}(u),F_A+(\mu\circ u)\omega_\Sigma\big).$$

Heuristically, $\mathcal{E}$ is an infinite dimensional vector bundle over $\mathcal{B}^*$, and $\mathcal{S}$ is a smooth section of $\mathcal{E}$. The zero set $\mathcal{S}^{-1}(0)\subseteq\mathcal{B}^*$ consists of all vortex classes over Σ. Assume that $W\in\mathcal{S}^{-1}(0)$. Then formally, there is a canonical map $T:T_{(W,0)}\mathcal{E}\to\mathcal{E}_W$, where $\mathcal{E}_W\subseteq\mathcal{E}$ denotes the fiber over W. We define the vertical differential of $\mathcal{S}$ at W to be the map

$$d^V\mathcal{S}(W)=T\,d\mathcal{S}(W):T_W\mathcal{B}^*\to\mathcal{E}_W. \tag{1.31}$$

[32] Here $\Gamma(E)$ denotes the space of smooth sections of a vector bundle $E\to\Sigma$.

Heuristically, if this map is Fredholm and surjective, for every $W \in \mathcal{S}^{-1}(0)$, then the zero set $\mathcal{S}^{-1}(0)$ is a smooth submanifold of $\mathcal{B}^*$. The dimension of a connected component of this submanifold equals the Fredholm index of $d^V\mathcal{S}(W)$, where W is any point in the connected component.

At a formal level, in the case $\Sigma = \mathbb{C}$, equipped with $\omega_\Sigma = \omega_0$, the vertical differential (1.31) coincides with the operator $\mathcal{D}_W^{p,\lambda}$, which was defined in (1.25,1.26) and occurred in the Fredholm result, Theorem 4. To see this, let $W \in \mathcal{B}^*$. We interpret $T_W\mathcal{B}^*$ as a quotient, as follows. Let P be a smooth G-bundle over Σ, and $(A,u) \in \widetilde{\mathcal{B}}^*_P$. Denoting $w := (P,A,u)$, the infinitesimal action at the point (A,u), corresponding to the action of $\mathcal{G}_P$ on $\widetilde{\mathcal{B}}^*_P$, is given by

$$L_w : \mathrm{Lie}(\mathcal{G}_P) = \Gamma(\mathfrak{g}_P) \to T_{(A,u)}\widetilde{\mathcal{B}}^*_P = \Gamma\left(A^1(\mathfrak{g}_P) \oplus TM^u\right), \quad L_w\xi = (-d_A\xi, L_u\xi),$$

where $d_A\xi := d\xi + [A,\xi]$. Defining

$$\widetilde{\mathcal{X}}_w := T_{(A,u)}\widetilde{\mathcal{B}}^*_P/\mathrm{im}L_w, \tag{1.32}$$

we may identify

$$T_W\mathcal{B}^* = \mathcal{X}_W := \left(\coprod_{w\in W} \widetilde{\mathcal{X}}_w\right)/\sim, \tag{1.33}$$

where $\sim$ denotes the natural lift of the equivalence relation on $\widetilde{\mathcal{B}}^*$. Assume formally that $\widetilde{\mathcal{B}}^*_P$ and $\mathrm{Lie}(\mathcal{G}_P)$ are equipped with a $\mathcal{G}_P$-invariant Riemannian metric and a $\mathcal{G}_P$-invariant inner product, respectively. For $(A,u) \in \widetilde{\mathcal{B}}^*_P$ we denote by $L_w^* : T_{(A,u)}\widetilde{\mathcal{B}}^*_P \to \mathrm{Lie}(\mathcal{G}_P)$ the adjoint map of L_w. Then by (1.32), we may identify

$$\widetilde{\mathcal{X}}_w = \ker L_w^* \subseteq \Gamma\left(A^1(\mathfrak{g}_P) \oplus TM^u\right).$$

Using this and (1.33,1.30), the vertical differential (1.31) at $W \in \mathcal{S}^{-1}(0)$ agrees with the map

$$\left(\coprod_{w\in W} \ker L_w^*\right)/\sim \;\to\; \left(\coprod_{w\in W} \Gamma\left(A^{0,1}(TM^u) \oplus A^2(\mathfrak{g}_P)\right)\right)/\sim,$$

given by (1.26), in the case $\Sigma = \mathbb{C}$ and $\omega_\Sigma = \omega_0$. Here on either side, $\sim$ denotes a natural lift of the equivalence relation on $\widetilde{\mathcal{B}}^*$.

1.5. Remarks, related work, organization, and acknowledgments

Remarks.

5. Remark (Vortices as triples). In some earlier work (e.g. [**CGS**] and [**Zi1**]), the G-bundle P was fixed and the vortex equations were seen as equations for a pair (A,u) rather than a triple (P,A,u).[33] The motivation for making P part of the data is twofold:

When formulating convergence for a sequence of vortex classes over $\mathbb{C}$ to a stable map, one has to pull back the vortices by translations of $\mathbb{C}$. (See Section 2.2.) If the principal bundle is fixed and vortices are defined as pairs (A,u) solving (1.8,1.9), then there is no natural such pullback.[34] However, there *is* a natural

[33] However, in [**MT**] I. Mundet i Riera and G. Tian took the viewpoint of the present memoir.

[34] Given a G-bundle P over $\mathbb{C}$, we may of course choose a trivialization of P, and then define a pull back for pairs (A,u), using the trivialization. However, this approach is unnatural, since it depends on the choice of a trivialization.

pullback if the bundle is made part of the data for a vortex. More generally, it is possible to pull back vortex triples (P, A, u) by a Kähler transformation of a Riemann surface equipped with a compatible area form.

Another motivation is the following: If the area form or the complex structure on the surface Σ vary, then in the limit we may obtain a surface Σ' with singularities. It does not make sense to consider P as a bundle over Σ'. One way of solving this problem is by decomposing Σ' into smooth surfaces, and constructing smooth G-bundles over these surfaces. Hence the G-bundle should be viewed as a varying object.

Once P is made part of the data, it is natural to consider *equivalence classes* of triples $(P, A, u) \in \widetilde{\mathcal{M}}$ (as defined in (1.15)), rather than the triples themselves. One reason is that all important quantities, like energy density and energy, are invariant (or equivariant) under equivalence. Note also that the bubbling and Fredholm results are more naturally stated for equivalence classes of vortices. Viewing the equivalence classes as the fundamental objects also matches the physical viewpoint that the "gauge field", i.e., the connection A, is physically relevant only "up to gauge". □

6. Remark. Let Σ be the plane $\mathbb{C}$, equipped with the standard area form ω_0, and consider the trivial G-bundle $P_0 := \mathbb{C} \times G$. Then the solutions (A, u) of the vortex equations (1.8,1.9) on P_0 bijectively correspond to solutions $(\Phi, \Psi, f) \in C^\infty\big(\mathbb{C}, \mathfrak{g} \times \mathfrak{g} \times M\big)$ of the equations

$$\partial_s f + L_f\Phi + J(f)(\partial_t f + L_f\Psi) = 0, \tag{1.34}$$

$$\partial_s\Psi - \partial_t\Phi + [\Phi, \Psi] + \mu(f) = 0. \tag{1.35}$$

Here we denote by s and t the standard coordinates in $\mathbb{C} = \mathbb{R}^2$, and in the second equation we identify the Lie algebra $\mathfrak{g}$ with its dual via the inner product $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$. The correspondence maps such a triple (Φ, Ψ, f) to $\big(A, u\big)$, where A denotes the connection on P_0 defined by

$$A_{(z,g)}(\zeta, g\xi) := \big(\Phi(z)ds + \Psi(z)dt\big)\zeta + \xi, \quad \forall \zeta \in T_z\mathbb{C},\ \xi \in \mathfrak{g},\ z \in \mathbb{C},\ g \in G,$$

and the map $u : P_0 \to M$ is given by $u(z, g) := g^{-1}f(z)$. The group $C^\infty(\mathbb{C}, G)$ acts on the set $\widetilde{\mathcal{M}}_0$ of solutions of (1.34,1.35) by

$$h^*(\Phi, \Psi, f) := \Big(h^{-1}\partial_s h + \mathrm{Ad}_{h^{-1}}\Phi, h^{-1}\partial_t h + \mathrm{Ad}_{h^{-1}}\Psi, h^{-1}f\Big),$$

where we denote the adjoint representation of an element $g \in G$ by $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$. This group naturally corresponds to the gauge group $C^\infty_G(P_0, G)$, and its action to the action

$$g^*(A, u) := \big(g^{-1}dg + \mathrm{Ad}_{g^{-1}}A, g^{-1}u\big).$$

Since every G-bundle over $\mathbb{C}$ is trivializable, it follows that the quotient of $\widetilde{\mathcal{M}}_0$ by the action of $C^\infty(\mathbb{C}, G)$ bijectively corresponds to the quotient $\mathcal{M} = \widetilde{\mathcal{M}}/\sim$, consisting of gauge equivalence classes of triples (P, A, u) of solutions of (1.8,1.9). Hence the results of the present memoir can alternatively be formulated in terms of solutions of the equations (1.34,1.35). However, the intrinsic approach using equations (1.8,1.9) seems more natural. □

7. Remark (Asphericity). Without the asphericity condition one needs to include holomorphic spheres in the fibers of the Borel construction in the definition of a stable map. In this situation, to compactify the space of vortices over $\mathbb{C}$ with

an upper energy bound, one needs to combine the proof of Theorem 3 with the analysis carried out by I. Mundet i Riera and G. Tian in [**Mu1**, **MT**], or by A. Ott in [**Ott**]. □

8. Remark (Quotient spaces). The space $\mathcal{X}_W^{p,\lambda}$ occurring in Theorem 4 is a quotient of a disjoint union of normed vector spaces. It is canonically isomorphic to the space $\widetilde{\mathcal{X}}_w^{p,\lambda}$, for every representative w of W. Similar statements hold for $\mathcal{Y}_W^{p,\lambda}$. The description of the spaces $\mathcal{X}_W^{p,\lambda}$ and $\mathcal{Y}_W^{p,\lambda}$ as such quotients may look unconventional, however, it seems natural, since it does not involve any choice of a representative of W.

Alternatively, one could phrase the Fredholm result in terms of the spaces $\widetilde{\mathcal{X}}_w^{p,\lambda}$ and $\widetilde{\mathcal{Y}}_w^{p,\lambda}$. However, in view of the last part of Remark 5, this seems less natural than the present formulation. □

9. Remark (Decay condition and vortices). The condition $\|\sqrt{e_w}\|_{p,\lambda} < \infty$ in the definition (1.16) of $\widetilde{\mathcal{B}}_\lambda^p$ and the requirement $1 - 2/p < \lambda < 2 - 2/p$ in Theorem 4(ii) capture the geometry of finite energy vortices over $\mathbb{C}$, in the following sense. Let $w = (P, A, u) \in \widetilde{\mathcal{B}}_{\mathrm{loc}}^p$ be a finite energy vortex such that $u(P)$ has compact closure. (Here $\widetilde{\mathcal{B}}_{\mathrm{loc}}^p$ is defined as at the beginning of Section 1.4.) Then for every $\varepsilon > 0$ there exists a constant C such that $e_w(z) \le C|z|^{-4+\varepsilon}$, for every $z \in \mathbb{C} \setminus B_1$. This follows from Theorem 99 in Appendix A.5 and [**Zi2**, Corollary 1.4].

It follows that $w \in \widetilde{\mathcal{B}}_\lambda^p$ if $\lambda < 2 - 2/p$. This bound is sharp. To see this, let $\lambda \ge 2 - 2/p$ and $M := S^2$, equipped with the standard symplectic form ω_{st}, complex structure $J := i$, and the action of the trivial group $G := \{\mathbf{1}\}$. Consider the inclusion $u : \mathbb{C} \times \{\mathbf{1}\} \cong \mathbb{C} \to S^2 \cong \mathbb{C} \cup \{\infty\}$. Since this map is holomorphic, the triple $\big(\mathbb{C} \times \{\mathbf{1}\}, 0, u\big)$ is a finite energy vortex whose image has compact closure. It does not lie in $\widetilde{\mathcal{B}}_\lambda^p$.

On the other hand, every $w \in \widetilde{\mathcal{B}}_\lambda^p$ has finite energy whenever $p > 2$ and $\lambda > 1 - 2/p$.[35] The latter condition is sharp. To see this, consider $M := \mathbb{R}^2, \omega := \omega_0, G := \{\mathbf{1}\}, J := i$. We choose a smooth map $u : \mathbb{C} \times \{\mathbf{1}\} \cong \mathbb{C} \to \mathbb{R}^2$, such that

$$u(z) = \Big(\cos\Big(\sqrt{\log|z|}\Big), \sin\Big(\sqrt{\log|z|}\Big)\Big), \quad \forall z \in \mathbb{C} \setminus B_2.$$

Then the triple $\big(\mathbb{C} \times \{\mathbf{1}\}, 0, u\big)$ lies in $\widetilde{\mathcal{B}}_\lambda^p$ for every $p > 2$ and $\lambda \le 1 - 2/p$. However, it has infinite energy.[36] □

10. Remark (Index). The condition $\lambda < 2 - 2/p$ in part (ii) of Theorem 4 is needed for the map $\mathcal{D}_W^{p,\lambda}$ to have the right Fredholm index. Namely, let $\lambda > 1 - 2/p$ be such that $\lambda + 2/p \notin \mathbb{Z}$, and $W \in \mathcal{B}_\lambda^p$. Then the proof of Theorem 4 shows that $\mathcal{D}_W^{p,\lambda}$ is Fredholm with index equal to

$$(2 - k)(\dim M - 2\dim G) + 2\big\langle c_1^G(M, \omega), [W]\big\rangle,$$

[35] This follows from the estimates

$$\|\sqrt{e_w}\|_2 \le \big\|\sqrt{e_w}\langle\cdot\rangle^\lambda\big\|_p \big\|\langle\cdot\rangle^{-\lambda}\big\|_q, \quad \big\|\langle\cdot\rangle^{-\lambda}\big\|_q < \infty,$$

where $q := 2p/(p - 2)$. The first estimate is Hölder's inequality and the second one follows from a calculation in radial coordinates.

[36] In the present setting, a simpler example of an infinite energy triple $w = (P, A, u)$ satisfying $\sqrt{e_w} \in L_\lambda^p$ for every $p > 2$ and $\lambda \le 1 - 2/p$, is $w := \big(\mathbb{C} \times \{\mathbf{1}\}, 0, u\big)$, where $u(z) := \Big(\sqrt{\log|z|}, 0\Big)$, for every $z \in \mathbb{C} \setminus B_2$. However, the closure of the image of such a map u is non-compact, since it contains the set $[1, \infty) \times \{0\}$. Therefore, w does not lie in $\widetilde{\mathcal{B}}_\lambda^p$ for any p and λ.